DEL PEZZO SURFACES IN WEIGHTED PROJECTIVE SPACES

ERIK PAEMURRU

Abstract. We study singular del Pezzo surfaces that are quasi-smooth and well-formed weighted hypersurfaces. We give an algorithm how to classify all of them.

1. Introduction.

A basic object of study in Algebraic Geometry is an algebraic variety. The simplest algebraic varieties are those that are covered by rational curves, i.e. rationally connected ones. Varieties with a positive first Chern class, the so-called Fano varieties (see [28]), are the building blocks of rationally connected varieties. They have been studied for a long time and have often been used to produce counterexamples to old-standing conjectures (see [12], [27], [3], [52]).

Classically, Fano varieties were assumed to be smooth. However, during the last decades the progress in the area — including the development of the Minimal Model Program in the works of Birkar, Corti, Hacon, Kawamata, McKernan, Mori, Shokurov, (see [46], [13], [36], [32], [7]) and others — both gave tools and posed problems dealing with mildly singular Fano varieties. Unfortunately, singular Fano varieties do not form a bounded family even in dimension two, and their classification seems to be absolutely hopeless in higher-dimensions. Nevertheless, we know many partial classification-type results about singular del Pezzo surfaces thanks to combined efforts of many algebraic geometers (see [58], [24], [40], [59], [34], [33], [47], [37], [1], [5], [11], [23], [30]).

A central theme in differential geometry is to try and characterize a given geometric structure by a metric with the best curvature properties. In general, a metric can not always be characterized only by its curvature properties, since even the number of degrees of freedom may not match. However, there is such a natural question in Kähler geometry: for a given compact complex Kähler manifold, determine whether it admits a Kähler–Einstein metric. That is, a Kähler metric whose Ricci curvature is proportional to the metric tensor. This problem, known as the Calabi problem, has a meaning only for complex manifolds with negative, vanishing, or positive first Chern class.

For complex manifolds with negative or vanishing first Chern class, the Calabi problem was solved by Yau and Aubin (see [4], [55] and . [56]). The problem of the existence of Kähler-Einstein metrics on compact complex manifolds with positive first Chern class, i.e. Fano manifolds, is a very subtle problem that still remains unsolved (see [56], [16], [19], [41]). For two-dimensional Fano manifolds, i.e. for smooth del Pezzo surfaces, the Calabi problem has been completely solved by Tian and Yau (see [51], [53]). For smooth toric Fano manifolds this problem has been solved by Wang and Zhu (see [54]).

For Fano manifolds, we know many obstructions to the existence of a Kähler–Einstein metric that are due to Matsushima, Lübke, Futaki, Tian, Donaldson, Ross, Thomas, Gauntlett, Martelli, Sparks and Yau (see [39], [38], [21], [52], [15], [42], [43], [22]). Moreover, it has been conjectured by Yau, Tian and Donaldson that a Fano manifold admits a Kähler–Einstein metric if and only if it is stable in a certain sense (see [43, Conjecture 2.8], [41]). Proving this conjecture is currently a major research programme in geometry. One direction of this conjecture is now almost proved by Donaldson (see [17], [18]).

The existence of an orbifold Kähler–Einstein metric on a Fano orbifold X is equivalent to the existence of a solution to the global complex Monge–Ampere equation on X. This problem remains out of reach even in dimension two. We know many obstructions to the existence of such an orbifold metric (see [44], [45], [22], [49]). However, del Pezzo surfaces with quotient singularities are very far from being classified. Mostly because of this, the Calabi problem for them is very far from being solved. So, it seems natural to impose more restrictions on

The author was supported by a grant from the William Manson bequest of the University of Edinburgh.

the class of singular del Pezzo surfaces under consideration, e.g. to consider only singular del Pezzo surfaces that are quasi-smooth and well-formed (see [25, Definition 6.9]) hypersurfaces in weighted projective spaces.

Let S_d be a hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d, where $1 \leqslant a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$ are some natural numbers. Then S_d is given by

$$\phi(x, y, z, t) = 0 \subset \mathbb{P}\big(a_0, a_1, a_2, a_3\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

where $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, and ϕ is a quasi-homogeneous polynomial of degree d with respect to these weights. The equation

$$\phi(x, y, z, t) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

defines a three-dimensional hypersurface quasi-homogeneous singularity (V, O), where $O = (0, 0, 0, 0)$. Recall that S_d is called *quasi-smooth* if the singularity (V, O) is isolated. Recall that S_d is called *well-formed* if

$$\gcd(a_1, a_2, a_3) = \gcd(a_0, a_2, a_3) = \gcd(a_0, a_1, a_3) = \gcd(a_0, a_1, a_2) = 1,$$

and each positive integer $\gcd(a_0, a_1)$, $\gcd(a_0, a_2)$, $\gcd(a_0, a_3)$, $\gcd(a_1, a_2)$, $\gcd(a_1, a_3)$, $\gcd(a_2, a_3)$ divides d. If the hypersurface S_d is quasi-smooth and well-formed, then it follows from [35, Theorem 7.9], [35, Proposition 8.13], [35, Remark 8.14.1], [35, Theorem 11.1], and the adjunction formula that the following conditions are equivalent

- the inequality $d < a_0 + a_1 + a_2 + a_3$ holds,
- the singularity (V, O) is a rational singularity,
- the singularity (V, O) is a Kawamata log terminal singularity,
- the hypersurface S_d is a del Pezzo surface with quotient singularities.

Starting from now, suppose that $d < \sum_{i=0}^{n} a_i$ and that the hypersurface S_d is quasi-smooth and well-formed. Put $I = a_0 + a_1 + a_2 + a_3 - d$. Recall that I is usually called the index of the hypersurface $S_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$. For every I we have infinitely many possibilities for the sextuple $(a_0, a_1, a_2, a_3, d, I)$ such that there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$. This is not surprising, since we know there are infinitely many families of del Pezzo surfaces with quotient singularities.

Problem 1.1. *Describe* all sextuples $(a_0, a_1, a_2, a_3, d, I)$ such that there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$.

This problem was posed by Orlov long time ago in order to test his conjecture about the existence of a full exceptional collection on del Pezzo surfaces with quotient singularities. Later his conjecture was proved by Kawamata, Elagin, Ishii, and Ueda (see [31], [20], [26]).

The first step in solving Problem 1.1 was done by Johnson and Kollár who proved the following:

Theorem 1.2 ([29, Theorem 8]). *Suppose that $I = 1$. Then*

- *either $(a_0, a_1, a_2, a_3, d) = (2, 2m + 1, 2m + 1, 4m + 1, 8m + 4)$ for some $m \in \mathbb{N}$,*
- *or the quintuple (a_0, a_1, a_2, a_3, d) lies in the sporadic set*

$$\left\{\begin{array}{l}(1,1,1,1,3),(1,1,1,2,4),(1,1,2,3,6),(1,2,3,5,10),(1,3,5,7,15),(1,3,5,8,16),\\(2,3,5,9,18),(3,3,5,5,15),(3,5,7,11,25),(3,5,7,14,28),(3,5,11,18,36),\\(5,14,17,21,56),(5,19,27,31,81),(5,19,27,50,100),(7,11,27,37,81),\\(7,11,27,44,88),(9,15,17,20,60),(9,15,23,23,69),(11,29,39,49,127),\\(11,49,69,128,256),(13,23,35,57,127),(13,35,81,128,256)\end{array}\right\}.$$

Moreover, for each listed quintuple (a_0, a_1, a_2, a_3, d), there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d = a_0 + a_1 + a_2 + a_3 - 1$.

The second step in solving Problem 1.1 was done by Cheltsov and Shramov who solved Problem 1.1 for $I = 2$ (see [11, Corollary 1.13]).

For Cheltsov, Johnson, Kollár, and Shramov, the main motivation to prove Theorem 1.2 was the Calabi problem for del Pezzo surfaces with quotient singularities and, in particular,

the Calabi problem for quasi-smooth well-formed hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$. Regarding the latter, Gauntlett, Martelli, Sparks, and Yau proved

Theorem 1.3 ([22]). *The surface S_d does not admit an orbifold Kähler–Einstein metric if $I > 3a_0$.*

Thus, the Calabi problem for quasi-smooth well-formed hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ has negative solution if $I > 3a_0$. On the other hand, Araujo, Boyer, Demailly, Galicki, Johnson, Kollár, and Nakamaye proved that the Calabi problem for quasi-smooth well-formed hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d = a_0 + a_1 + a_2 + a_3 - 1$ almost always has positive solution.

Theorem 1.4 ([14], [29], [2], [6]). *Suppose that $I = 1$. Then S_d admits an orbifold Kähler–Einstein metric except possibly the case when $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the polynomial $\phi(x_0, x_1, x_2, x_3)$ does not contain the monomial $x_1x_2x_3$.*

The proof of Theorem 1.4 implicitly uses the so-called α-invariant introduced by Tian for smooth Fano varieties in [50]. For S_d, its algebraic counterpart can be defined as

$$\alpha(S_d) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the log pair } \Big(S_d, \lambda D\Big) \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D \equiv -K_{S_d}\end{array}\right\},$$

and one can easily extend this definition to any Fano variety with at most Kawamata log terminal singularities. Tian, Demailly, and Kollár showed that the α-invariant plays an important role in the existence of an orbifold Kähler–Einstein metric on Fano varieties with quotient singularities (see [50], [14], [48], [8], [10, Theorem A.3]). In particular, we have

Theorem 1.5 ([50], [14], [10, Theorem A.3]). *If $\alpha(S_d) > 2/3$, then S_d admits an orbifold Kähler–Einstein metric.*

Araujo, Boyer, Demailly, Galicki, Johnson, Kollár, and Nakamaye proved that $\alpha(S_d) > 2/3$ if $I = 1$ except exactly one case when $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the polynomial $\phi(x_0, x_1, x_2, x_3)$ does not contain the monomial $x_1x_2x_3$ (in this case $\alpha(S_d) = 7/10 < 2/3$ by [9]). A similar approach was used by Boyer, Cheltsov, Galicki, Nakamaye, Park, and Shramov for $I \geqslant 2$ (see [6], [9], [11]).

It is seems unlikely that Problem 1.1 has a *nice* solution for all I at once. However, the results by Cheltsov, Johnson, Kollár, and Shramov indicate it seems possible to solve Problem 1.1 for any fixed I. The main purpose of this paper is to prove this and to give an algorithm that solves Problem 1.1 for any fixed I. The result of our classification will be a set of quintuples and series of quintuples in the form (a_0, a_1, a_2, a_3, d), where a_0, a_1, a_2, a_3 are the ordered weights, and $d = a_0 + a_1 + a_2 + a_3 - I$ is the degree.

We hope that our classification can be useful to produce vast number of examples of non-Kähler–Einstein del Pezzo surfaces with quotient singularities using different kind of existing obstructions. For example, recently Spotti proved

Theorem 1.6 ([49]). *Let S be a del Pezzo surfaces with at most quotient singularities, and let N is the biggest natural number such that S_d has a quotient singularity $\mathbb{C}^2/G$ with $N = |G|$, where G is a finite subgroup in $\mathrm{GL}_2(\mathbb{C})$ that does not contain quasi-reflections. Then S does not admit an orbifold Kähler–Einstein metric if $K_S^2 N \geqslant 12$.*

Using our classification, we immediately obtain a huge number of examples of quasi-smooth well-formed hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ that do not admit an orbifold Kähler–Einstein metric by Theorem 1.6 such that the obstruction found by Gauntlett, Martelli, Sparks, and Yau, i.e. Theorem 1.3, is not applicable. Examples can easily be drawn from two-parameter series of our classification, when the first weight is large enough. Such series exist for any $I \geqslant 2$. We will give four tuples (a_0, a_1, a_2, a_3, d), which will show that the theorems 1.3 and 1.6 are independent.

- $(1, 3, 4, 8, 12)$, from the series $(1, 3, 3a + 1, 3b + 2, 3a + 3b + 3)$ with $I = 4$, satisfies both Theorem 1.3 ($4 = I > 3a_0 = 3$) and Spotti's inequality ($16 = K_S^2 N \geqslant 12$).
- $(1, 3, 7, 8, 15)$, from the series $(1, 3, 3a + 1, 3b + 2, 3a + 3b + 3)$ with $I = 4$, satisfies Theorem 1.3 ($4 = I > 3a_0 = 3$) but not Spotti's inequality ($11\frac{3}{7} = K_S^2 N \ngeqslant 12$).

- $(2,2,3,7,10)$, from the series $(2,2,2a+1,2b+1,2a+2b+2)$ with $I=4$, does not satisfy Theorem 1.3 ($4=I\not\geqslant 3a_0=6$) but does satisfy Spotti's inequality ($13\frac{1}{3}=K_S^2N\geqslant 12$).
- $(2,2,3,3,6)$, from the series $(2,2,2a+1,2b+1,2a+2b+2)$ with $I=4$, does not satisfy Theorem 1.3 ($4=I\not\geqslant 3a_0=6$) nor Spotti's inequality ($8=K_S^2N\not\geqslant 12$).

These examples show that the previous inequality by Gauntlett, Martelli, Sparks, and Yau (Theorem 1.3) is independent of the new inequality discovered by Spotti (Theorem 1.6). Thus, Spotti's inequality is really a new and powerful obstruction to the existence of orbifold Kähler–Einstein metrics on del Pezzo surfaces with quotient singularities.

Let us describe the structure of the paper. At the start of Section 2, we will introduce the theorems we will be using, and some basic terms necessary for the solution. At first, it will not be immediately clear to the reader where the precise form of the definitions comes from. The results from later chapters help to understand them better. In Section 3, we will show some structure in the solution tuples (a_0,a_1,a_2,a_3,d), to make it easier to handle the problem. We will use this structure in Section 4 to show two powerful results. First, that it suffices to consider a set of conditions weaker than the natural smoothness conditions. This result, given in Proposition 4.1, immediately makes the problem much simpler, along with the final classification algorithm. Secondly, we will show in Theorem 4.2 that the solution forms a set of infinite series. This effectively gives us the classification. We will leave the proofs of the longer propositions to Paragraph 5, since the results of the propositions are much more important and enlightening than the proofs. At the end, we give the classification for small index cases $I=1,2,\dots,6$, in Section 6. The algorithm has been programmed by the author, a source code sample for the main idea of the algorithm is given in Appendix B.

This paper provides the algorithm to find the answer to Orlov's problem, Problem 1.1, for any fixed I, as well as the general form of the answer for any I. The surprisingly rigid form of the answer, given in detail in the definition of the series in Definition 2.5, gives us the power of drawing conclusions about the hypersurfaces for all I at once, without explicitly calculating them. The program that calculates the classification of the hypersurfaces for any given I is available from the author.

Acknowledgements. I would like to thank the William Manson bequest, which supported the project financially, the School of Mathematics of the University of Edinburgh, and my project supervisor Professor Ivan Cheltsov for giving me the opportunity to work on this project. The experience and knowledge gained through this work will undoubtedly be of great benefit to me in the near future.

2. Approaching the problem.

We will classify hypersurfaces, given the Fano index I. We are searching for the set of quintuples that satisfy the following theorem ([29, Conditions 2.1, 2.2 and 2.3]).

Theorem 2.1. *The non-degenerate hypersurface is well-formed and quasi-smooth iff all of the following conditions hold:*

(i-ii) *The hypersurface is well-formed.*

(iii) *The hypersurface is not degenerate, that is, for all weights a_i, we have $d\neq a_i$.*

(iv) *For every i, there exists j (j may equal i) such that there exists a monomial $x_i^{m_i}x_j$ of degree d, where $m_i\geqslant 1$.*

(v) *If $i<j$ and $\gcd(a_i,a_j)>1$ then there is a monomial $x_i^{b_i}x_j^{b_j}$ of degree d, where $b_i,b_j\geqslant 0$ and $b_i+b_j\geqslant 2$.*

(vi) *For every $i<j$, either there is a monomial $x_i^{b_i}x_j^{b_j}$ of degree d, or there exist $k<l$ such that the indices i,j,k,l are pairwise different and there are monomials $x_i^{c_i}x_j^{c_j}x_k$ and $x_i^{d_i}x_j^{d_j}x_l$ of degree d, where $b_i,b_j,c_i,c_j,d_i,d_j\geqslant 0$ and $b_i+b_j\geqslant 2$ and $c_i+c_j\geqslant 1$ and $d_i+d_j\geqslant 1$.*

We will translate this theorem into number theory, and use the following theorem instead, adding the results from [11, Theorem 2.3] and [11, Definitions 1.10 and 2.2].

Theorem 2.2. *The non-degenerate hypersurface is well-formed and quasi-smooth iff all of the following conditions hold:*

(i) $\gcd(a_0, ..., \widehat{a_i}, ..., \widehat{a_j}, ..., a_3) \mid d$
(ii) $\gcd(a_0, ..., \widehat{a_i}, ..., a_3) = 1$
(iii) $d > a_3$
(iv) *For every* i, *there exists* j *(*j *may equal* i*), such that* $d - a_j \vdots a_i$.
(v) *If* $i < j$ *and* $\gcd(a_i, a_j) > 1$ *then one of the following holds*
  – $d \vdots a_i$
  – $d \vdots a_j$
  – $d - a_j \vdots a_i$
  – $d - a_i \vdots a_j$
  – *there exists* $b_j \geqslant 2$ *such that* $d - a_j b_j \vdots a_i$ *and* $d - a_j b_j \geqslant 0$.
(vi) *For every* $i < j$, *(at least) one of the following holds:*
  – *One of the following holds:*
    ∗ $d \vdots a_i$
    ∗ $d \vdots a_j$
    ∗ $d - a_j \vdots a_i$
    ∗ $d - a_i \vdots a_j$
    ∗ *there exists* $b_j \geqslant 2$ *such that* $d - a_j b_j \vdots a_i$ *and* $d - a_j b_j \geqslant 0$.
  – *For pairwise different indices* i, j, k, l, *satisfying* $k < l$, *both of the following hold:*
    ∗ *One of the following holds:*
      · $d - a_k \vdots a_i$
      · $d - a_k \vdots a_j$
      · *there exists* $c_j \geqslant 1$ *such that* $d - a_k - a_j c_j \vdots a_i$ *and* $d - a_k - a_j c_j \geqslant 0$.
    ∗ *One of the following holds:*
      · $d - a_l \vdots a_i$
      · $d - a_l \vdots a_j$
      · *there exists* $d_j \geqslant 1$ *such that* $d - a_l - a_j d_j \vdots a_i$ *and* $d - a_l - a_j d_j \geqslant 0$.

Also, one of the following conditions must hold:

(Type-I) $I = a_i + a_j$, *for some* $i \neq j$
(Type-II) $I = a_i + \frac{a_j}{2}$, *for some* $i \neq j$
(Type-III) $\big(a_0, a_1, a_2, a_3, d\big) = \big(I - k, I + k, a, a + k, 2a + I + k\big)$, *where* $1 \leqslant k < I$ *and* $a \geqslant I + k$.
(Type-IV) $\big(a_0, a_1, a_2, a_3, d\big)$ *belongs to one of the infinite series listed in Table 18 or to the sporadic set given in Table 19 of the appendix.*

Tuples of Type-IV will simply be read from tables, so we will disregards these till the end of this paper. When we refer to condition (x), then we refer to condition (x) of Theorem 2.2. When we refer to Type-X, we refer to Type-X of Theorem 2.2. When speaking of a tuple, we always assume it is of Type I, II or III, and that it is ordered, with a strict inequality $d > a_3$ for the degree. If a tuple satisfies the Theorem 2.2 (disregarding Type-IV) we will call it **valid**. The aim of this paper is to find all the valid tuples, given a Fano index I. At first it is somewhat easier to deal with a sub-case of the valid tuples, when only conditions (i)-(iv) are satisfied (and the tuple is of some Type I-III). We will call such tuples **solid**. It is clear that each valid tuple is solid. We will show in Proposition 4.1 that all solid tuples are also valid, and that these definitions are equivalent. For this, we must first find some structure in the tuples.

As a remark, Type-III of Theorem 2.2 was simplified from $0 \leqslant k < I$ in [11, Theorem 2.2] to $1 \leqslant k < I$. This is because the case $k = 0$ adds no valid tuples to our solution. This is very easy to show, we will prove it in the following proposition.

Proposition 2.3. *Having* $1 \leqslant k < I$ *in Type-III in Theorem 2.2 gives the same classification as having* $0 \leqslant k < I$.

Proof. We will show that the only valid tuple of Type-III, when assuming $k = 0$, is already listed in the tables (Type-IV). Since $k = 0$, our tuple has the form $(I, I, a, a, 2a + I)$. Since a valid tuple must be well-formed, we must have $\gcd(a_2, a_3) = \gcd(a, a) = a \mid 2a + I = d \implies a \mid I \implies a \leqslant I$. Since the tuple is ordered, we find $a \geqslant I$, implying $a = I$. Using well-formedness again, we find $1 = \gcd(a_1, a_2, a_3) = \gcd(I, I, I) = I$. So the tuple is $(1, 1, 1, 1, 3)$. But this already exists in the tables – the first tuple, choosing $n = 1$, in Table 18. Therefore, we can safely assume $k \geqslant 1$. □

We will assign an infinite series to each tuple of a subset of the tuples of Types I-III, and later show that this subset indeed contains all the tuples satisfying the Theorem 2.2. The tuples with an infinite series will be called **colourful**. Each colourful tuple will have a unique **class** number associated to it. The uniqueness is easy to check.

Definition 2.4. A tuple is colourful iff it satisfies one of the following:

- $I = a_0 + a_1$ – class 1
- $I = a_0 + a_2$ and $I > a_0 + a_1$ – class 2
- $I = a_1 + a_2$ and $I > a_0 + a_2$ – class 3
- $I = a_0 + \frac{a_1}{2}$ – class 4
- $I = \frac{a_0}{2} + a_1$ and $I > a_0 + \frac{a_1}{2}$ – class 5
- The tuples is of Type-III, that is, it satisfies $\big(a_0, a_1, a_2, a_3, d\big) = \big(I - k, I + k, a, a + k, 2a + I + k\big)$, where $1 \leqslant k < I$ and $a \geqslant I + k$ – class 6

Next, we will identify each colourful tuple with an infinite series.

Definition 2.5. Infinite series for colourful tuples.

- For class 1 tuples, the corresponding series is $(a_0, a_1, a_2+xm, a_3+ym, a_2+a_3+(x+y)m)$, where $m = \operatorname{lcm}(a_0, a_1)$ and $x, y \in \mathbb{Z}$ are parameters such that the tuple is ordered.
- For class 2 tuples, the corresponding series is $(a_0, a_1, a_2, a_3 + xm, a_1 + a_3 + xm)$, where $m = \operatorname{lcm}(a_0, a_1, a_2)$ and $x \in \mathbb{Z}$ is a parameter.
- For class 3 tuples, the corresponding series is $(a_0, a_1, a_2, a_3 + xm, a_0 + a_3 + xm)$, where $m = \operatorname{lcm}(a_0, a_1, a_2)$ and $x \in \mathbb{Z}$ is a parameter.
- For class 4 tuples, the corresponding series is $(a_0, a_1, a_2+xm, \frac{a_1}{2}+a_2+xm, a_1+2a_2+2xm)$, where $m = \operatorname{lcm}(a_0, a_1)$ and $x \in \mathbb{Z}$ is a parameter.
- For class 5 tuples, the corresponding series is $(a_0, a_1, a_2+xm, \frac{a_0}{2}+a_2+xm, a_0+2a_2+2xm)$, where $m = \operatorname{lcm}(a_0, a_1)$ and $x \in \mathbb{Z}$ is a parameter.
- For class 6 tuples, the corresponding series is $\big(a_0, a_1, a_2 + xm, a_3 + xm, d + 2xm\big) = \big(I - k, I+k, a+xm, a+k+xm, 2a+I+k+2xm\big)$, where $m = \operatorname{lcm}(a_0, a_1, k) = \operatorname{lcm}(I-k, I+k, k)$ and $x \in \mathbb{Z}$ is a parameter.

All the parameters are bounded below in such a way that the tuple is ordered.

The purpose of these infinite series is the following: whenever a tuple in these infinite series is valid, all the ordered tuples in the infinite series are valid. This will be proved in Theorem 4.2. Now, we will introduce the term **defining weights**. Using these, it is easy to show there are only a finite number of infinite series for every index I.

Definition 2.6. The defining weights for the infinite series.

- For class 1, the weights a_0, a_1 are the class-defining weights, a_2, a_3 are the series-defining weights.
- For classes 2 and 3, the weights a_0, a_1, a_2 are the class-defining weights, a_3 is the series-defining weight.
- For classes 4,5 and 6, the weights a_0, a_1 are the class-defining weights, a_2 is the series-defining weight.

Let us now prove that there are only a finite number of infinite series for a fixed I.

Proposition 2.7. *There only a finite number of infinite series given in Definition 2.4, for any index I.*

Proof. It is easy to see there are only a finite number of choices for the class-defining weights. Since the tuple must be ordered, each of the class-defining weights must definitely be less than $2I$. Since they must all be positive, there are a finite number of choices.
Now, let us consider the series-defining weights. After specifying the class-defining weights, we can calculate m (Definition 2.5). This will be the least common multiple of all the class-defining weights, implying it is positive. We know that for any series-defining weights, if we add m to one of them, we still get the same series. So, since there are only one or two series-defining weights, and each of them can have only m different values modulo m, there are only a finite number of series-defining weights which define different series. Altogether, since there a finite number of

defining weights (counting both class- and series-defining weights), and each series is uniquely determined by the defining weights, there are a finite number of infinite series.

□

In practice, as there are infinitely many choices for equivalent series-defining weights, we will always choose the smallest possible ones such that the weights are ordered. Then, the parameters will be non-negative integers, instead of being strictly positive.

Next, we will show some structure in the solution, giving insight into the definitions of colourful tuples (2.4) and infinite series (2.5).

3. Structural Results.

In this section, we will try to find some structure in the solution. These results will justify the definition of infinite series in Definition 2.5, and will be stepping stones for more powerful theorems in Section 4.

We want to show that every valid tuple belongs to one of the infinite series in Definition 2.5. For this, we must show that every valid tuple is colourful, that is, that every valid tuple is of one the forms given in Definition 2.4. A slightly stronger result is proved in Proposition 3.3. It shows that every solid tuple is colourful, which implies it also for valid tuples. We will first prove two lemmas needed for this.

The following lemma reduces the amount of cases in Type II.

Lemma 3.1. *Let the ordered tuple* $\big(a_0, a_1, a_2, a_3, d\big)$ *with* $I = a_t + \frac{a_u}{2}$*, where one of* u *and* t *is equal to* 2 *and the other one less than* 2*, be solid (solid - conditions (i)-(iv) from Theorem 2.2 are satisfied). Then, either* $I = a_i + a_j$*, for some* $i, j \in \{0, 1, 2\}$ *with* $i \neq j$*, or* $I = a_0 + \frac{a_1}{2}$ *or* $I = a_1 + \frac{a_0}{2}$*.*

The proof is given in Lemma 5.1.
The following lemma removes the cases from Type I and Type II which were otherwise a potential source of sporadic solutions.

Lemma 3.2. *Let the ordered tuple* (a_0, a_1, a_2, a_3, d) *satisfying* $I = a_t + a_3$ *or* $I = \frac{a_t}{2} + a_3$ *or* $a_t + \frac{a_3}{2} = I$*, where* $t \in \{0, 1, 2\}$*, be solid. Then, either* $I = a_i + a_j$*, for some* $i, j \in \{0, 1, 2\}$ *with* $i \neq j$*, or* $I = a_0 + \frac{a_1}{2}$ *or* $I = a_1 + \frac{a_0}{2}$*.*

The proof is given in Lemma 5.2.
Now we can show the following result.

Proposition 3.3. *A solid tuple is colourful.*

Proof. We must show that whenever a tuple is solid, that is, has conditions (i)-(iv) satisfied, it has one of the forms described in Definition 2.4.

We will be using the fact that the tuple is either of Type-I, Type-II or Type-III. We will divide these into subclasses.

If we have a solid tuple, one of the following must hold:
1) $I = a_t + a_u$, for some $t, u \in \{0, 1, 2\}$, with $t \neq u$, or $I = a_0 + \frac{a_1}{2}$ or $I = \frac{a_0}{2} + a_1$;
2) The tuple is of the form $\big(I - k, I + k, a, a + k, 2a + I + k\big)$;
3) $I = a_t + a_3$ or $I = a_t + \frac{a_3}{2}$ or $I = \frac{a_t}{2} + a_3$ for some $t \in \{0, 1, 2\}$;
4) $I = a_t + \frac{a_u}{2}$, for some $t, u \in \{0, 1, 2\}$, with $t \neq u$, and either $t = 2$ or $u = 2$.

Points 1) and 2) are precisely those which define a colourful tuple. If 3) holds, then by Lemma 3.2 we must have that 1) also holds. If 4) holds, then by Lemma 3.1, we must have that 1) holds. So, in every case, either statement 1) or statement 2) must hold, showing that the tuple must be colourful.

□

The series in Definition 2.5 contain every colourful tuple of classes $1, 2, 3, 6$, implying they contain every valid tuple of classes $1, 2, 3, 6$ (Proposition 3.3). For classes 4 and 5 this is not immediately obvious, since the series insist extra conditions on the weight a_3. To show the series in Definition 2.5 do contain every valid tuple of classes $4, 5$, we must prove the following lemma. It proves a bit more – the series contain every solid tuple of classes 4 and 5. In other words, solid tuples of classes 4 and 5 have the form given in the definition of the series.

Lemma 3.4. *Let the ordered tuple (a_0, a_1, a_2, a_3, d) satisfying $a_t + \frac{a_u}{2} = I$, where $t, u \in \{0, 1\}$ and $t \neq u$, be solid. Then $a_3 = \frac{u}{2} + a_2$ and $d = 2a_3 = a_u + 2a_2$.*

The proof is given in Lemma 5.3.
We have shown that every solid tuple is indeed contained in the set of series given in Definition 2.5.

4. Main results.

The results from the previous section allow us to prove two powerful statements. First, that the weaker set of conditions we have been using so far guarantees quasi-smoothness. Secondly, that the solution forms infinite series.

The following proposition proves the equivalence of the definitions of solid and valid. Valid tuples are tuples, which satisfy the quasi-smoothness conditions. Solid tuples are tuples which satisfy a weaker set of conditions, namely conditions (i) to (iv) from Theorem 2.2. So, clearly, a valid tuple is solid. We will show the converse also holds.

Proposition 4.1. *A solid tuple is valid.*

The proof is given in Proposition 5.4.
So, to find valid tuples, it suffices only to consider conditions (i)-(iv) from the Theorem 2.2.
Next, we will show that whenever a tuple from a series from Definition 2.5 is valid, all the ordered tuples in that series are valid. Since every valid tuple must belong to one of these series, testing one tuple from each of the series enables us to find all the valid tuples. This is the main theorem of the paper.

Theorem 4.2. *Whenever a tuple is valid, all the ordered tuples in its corresponding series, given in Definition 2.5, are valid.*

The proof is given in Theorem 5.5.
This theorem, being the key to classifying the hypersurfaces, solves the problem of dealing with an infinite amount of tuples. For every I, each valid tuple belongs to a finite set of infinite series, given in Definition 2.5. Testing one tuple from each of those series shows whether all the tuples in that series are valid. After including the tuples from the tables (Type-IV of Theorem 2.2) to the valid infinite series, we get the classification.

5. Proofs.

In this section we give proofs to theorems in chapters 3 and 4. These are not particularly illuminating, and are proved using elementary number theory and combinatorics.

The following Lemmas 5.1 and 5.2 are proved using the following technique. We use condition (iv) of Theorem 2.2, always choosing the highest value for i, as this gives the strongest results. After every such use, we are able to eliminate one of the weights, starting from the highest – by "eliminate", we mean express it in terms of the smaller weights. After eliminating a weight, we try to express I in terms of the smaller weights, as required in the lemmas. If this is not successful, we will try to find a contradiction somewhere, for example, using that the weights must be ordered, or using well-formedness. If neither of these work, we use condition (iv) again, now choosing the next highest value for i. We try to prove as much as we can at the same time, meaning we deal with general $t, u, v \in \{0, 1, 2\}$ without specifying which is which. Sometimes this will not work, and we have to deal with all the 6 cases separately.

The following lemma reduces the amount of cases in Type II.

Lemma 5.1. *(Same as Lemma 3.1.) Let the ordered tuple $\big(a_0, a_1, a_2, a_3, d\big)$ with $I = a_t + \frac{a_u}{2}$, where one of u and t is equal to 2 and the other one less than 2, be solid (solid - conditions (i)-(iv) from Theorem 2.2 are satisfied). Then, either $I = a_i + a_j$, for some $i, j \in \{0, 1, 2\}$ with $i \neq j$, or $I = a_0 + \frac{a_1}{2}$ or $I = a_1 + \frac{a_0}{2}$.*

Proof. Let us define $w \in \{0, 1\}$ and $w \neq t, u$. Then $\{0, 1, 2\} = \{t, u, w\}$ and $d = a_w + \frac{a_u}{2} + a_3$. Let us use condition (iv) with $i = 3$. We have $d - a_j = a_w + \frac{a_u}{2} + a_3 - a_j \vdots a_3 \iff a_w + \frac{a_u}{2} - a_j \vdots a_3$. It is easy to see that $-a_3 < a_w + \frac{a_u}{2} - a_j < 2a_3$, therefore, $a_w + \frac{a_u}{2} - a_j \in \{0, a_3\}$, which is equivalent to $\frac{a_u}{2} \in \big\{a_j - a_w + x \mid x \in \{0, a_3\}\big\}$. If $j = u$ then $\frac{a_u}{2} \in \big\{a_w - x \mid x \in \{0, a_3\}\big\}$. We

get $x = 0$, giving $\frac{a_u}{2} = a_w$. We will use this result later. We have $j \neq w$, since otherwise we would have $a_u \in \{0, 2a_3\}$, which is false. Similarly, $(j, x) \neq (3, a_3)$.

We have $I = a_t + \frac{a_u}{2} \in \{a_t + a_j - a_w + x \mid x \in \{0, a_3\}\}$. If $j = u$, then $\frac{a_u}{2} = a_w$, as was shown above, and $I = a_t + \frac{a_u}{2} = a_t + a_w$. If $(j, x) = (t, 0)$, then $I = 2a_t - a_w$, and at the same time $a_u + a_w = 2a_j - a_w + 2x = 2a_t - a_w$. This combined gives $I = a_u + a_w$.
We must still look through the cases $(j, x) \in \{(3, 0), (t, a_3)\}$. For both of these cases, we have $d = a_w + \frac{a_u}{2} + a_3$, and $a_j = \frac{a_u}{2} + a_w - x$, the latter equation being the same as $x = \frac{a_u}{2} + a_w - a_j$. We also have $w \neq 2$, since we defined one of t, u to be equal to 2.

If $(j, x) = (3, 0)$ then $a_3 = \frac{a_u}{2} + a_w$ and $d = a_w + \frac{a_u}{2} + \frac{a_u}{2} + a_w = 2a_w + a_u = 2a_3$. Let us use condition (iv) with $i = 2$. It says $d - a_r = 2a_w + a_u - a_r \,\vdots\, a_2$. We have $d - a_r > 0$. Therefore, we must have $d - a_r = 2a_w + a_u - a_r \in \{a_2, 2a_2\}$. Let us look through these cases separately.

Let us first consider $r = 3$. Then $d - a_3 = a_3 = \frac{a_u}{2} + a_w \in \{a_2, 2a_2\}$. We can easily see that $\frac{a_u}{2} + a_w \neq 2a_2$, therefore $a_3 = a_2 = \frac{a_u}{2} + a_w$.
If $u = 2$, then $a_2 = 2a_w$. We have $I = a_t + \frac{a_2}{2} = a_t + a_w$, so our tuple is covered.
Since $u, w \neq 2$, we must have $t = 2$. We have $I = a_t + \frac{a_u}{2} = a_2 + \frac{a_u}{2} = a_3 + \frac{a_u}{2} = a_w + \frac{a_u}{2} + \frac{a_u}{2} = a_w + a_u$, so our tuple is covered.

Let us now consider $r = w$. Then $d - a_w = 2a_w + a_u - a_w = a_w + a_u$. Let us first consider $a_w + a_u < 2a_2$, then we must have $a_w + a_u = a_2$. But we have $a_3 = \frac{a_u}{2} + a_w < a_u + a_w = a_2$, which is a contradiction. Now let us consider $a_w + a_u = 2a_2$. We have $a_w = a_u = a_2$, which well-formedness requires to be equal to 1. But we have $I = a_t + \frac{a_u}{2}$, so $a_u \,\vdots\, 2$, which is a contradiction.

Next, let us consider $r = u$. Then $d - a_u = 2a_w + a_u - a_u = 2a_w \in \{a_2, 2a_2\}$.
We have $w \neq 2$. Let us consider $2a_w = 2a_2$. Well-formedness requires $\gcd(a_w, a_2) \mid d = 2a_3$. If an odd prime p divides a_2, it divides all of a_w, a_2, a_3. If 4 divides a_2, then 2 divides all of a_w, a_2, a_3. If $a_2 = 1$, then $a_u = 1$, which is a contradiction. Therefore, $a_u = a_w = a_2 = 2$, which is a contradiction.
Now, since $2a_w \neq 2a_2$, we must have $2a_w = a_2$. Well-formedness requires $\gcd(a_w, a_2) \mid d = 2a_3$. If an odd prime p divides a_2, it divides all of a_w, a_2, a_3. So there does not exist such p. We know that a_u is even. If 4 divides a_2, then 2 divides all of a_w, a_2, a_u, which is a contradiction. If $a_2 = 1$, then $a_u = 1$, which is a contradiction. Therefore, $a_2 = 2$, implying $a_w = 1$ and $a_u = 2$. We have $I = a_t + \frac{a_u}{2} = a_t + 1 = a_t + a_w$.

Finally, let us consider $r = t$. Then $d - a_t = 2a_w + a_u - a_t \in \{a_2, 2a_2\}$.
Let us first look at $2a_w + a_u - a_t = 2a_2$. If $t = 2$, then $2a_w + a_u = 3a_2$. This means $a_w = a_u = a_2$, which well-formedness requires to be 1. We have $a_u = 1$, which is a contradiction. Since $t \neq 2$, we must have $u = 2$. We get $2a_w - a_t = a_2$. We have $I = a_t + \frac{a_u}{2} = a_t + \frac{a_2}{2} = a_t + \frac{2a_w - a_t}{2} = a_t + a_w - \frac{a_t}{2} = a_w + \frac{a_t}{2}$, and $w, t \in \{0, 1\}$, so our tuple is covered.
Next, let us look at $2a_w + a_u - a_t = a_2$. If $u = 2$, then $2a_w - a_t = 0$, which means $2a_w = a_t$. Since $a_t > a_w$, we have $(w, t, u, 3) = (0, 1, 2, 3)$. This means $a_1 = 2a_0$ and $a_3 = \frac{a_u}{2} + a_w = \frac{a_2}{2} + a_0$. Our tuple becomes $\left(a_0, 2a_0, a_2, \frac{a_2}{2} + a_0\right)$. We must have $a_3 \geqslant a_2 \iff \frac{a_2}{2} + a_0 \geqslant a_2 \iff a_0 \geqslant \frac{a_2}{2} \iff 2a_0 \geqslant a_2$. We also must have $a_2 \geqslant a_1 \iff a_2 \geqslant 2a_0$. Therefore, $a_2 = 2a_0 = a_1$. We have $I = a_t + \frac{a_u}{2} = a_1 + \frac{a_2}{2} = a_1 + a_0$. So, our tuple is covered. This leaves the case $u \neq 2$, meaning $t = 2$. We have $2a_w + a_u = 2a_2$, which implies $a_2 = a_w + \frac{a_u}{2} = a_3$. We have $I = a_t + \frac{a_u}{2} = a_2 + \frac{a_u}{2} = a_w + \frac{a_u}{2} + \frac{a_u}{2} = a_w + a_u$, so our tuple is covered.

Lastly, we will look at the case $(j, x) = (t, a_3)$. We have $d = a_w + \frac{a_u}{2} + a_3$ and $x = a_w + \frac{a_u}{2} - a_j \iff a_3 = a_w + \frac{a_u}{2} - a_t$. Since $a_3 \geqslant a_u, a_w$, we must have $a_t < a_u, a_w$. Therefore, $t = 0$, which implies $u = 2$ and $w = 1$. We have $a_3 = -a_0 + a_1 + \frac{a_2}{2}$ and $d = a_1 + \frac{a_2}{2} + a_3 = a_1 + \frac{a_2}{2} - a_0 + a_1 + \frac{a_2}{2} = -a_0 + 2a_1 + a_2$. Our tuple is $\left(a_0, a_1, a_2, -a_0 + a_1 + \frac{a_2}{2}, -a_0 + 2a_1 + a_2\right)$ and $I = a_t + \frac{a_u}{2} = a_0 + \frac{a_2}{2}$.
Let us use condition (iv) with $i = 2$. We have $d - a_r = -a_0 + 2a_1 + a_2 - a_r \,\vdots\, a_2$. We can easily see $0 < d - a_r < 3a_2$. Therefore $-a_0 + 2a_1 + a_2 - a_r \in \{a_2, 2a_2\}$.

Let us first consider $r = 3$. Then, $d - a_3 = -a_0 + 2a_1 + a_2 + a_0 - a_1 - \frac{a_2}{2} = a_1 + \frac{a_2}{2} \in \{a_2, 2a_2\}$. We know $a_1 + \frac{a_2}{2} \leqslant \frac{3a_2}{2} < 2a_2$, so $a_1 + \frac{a_2}{2} = a_2$, which is equivalent to $\frac{a_2}{2} = a_1$. We have $I = a_0 + \frac{a_2}{2} = a_0 + a_1$, so our tuple is covered.

Next, let us consider $r = 2$. Then we have $d - a_2 = -a_0 + 2a_1 + a_2 - a_2 = -a_0 + 2a_1 \in \{a_2, 2a_2\}$. Since $-a_0 + 2a_1 < 2a_2$, we must have $-a_0 + 2a_1 = a_2$. We have $I = a_0 + \frac{a_2}{2} = a_0 + \frac{-a_0 + 2a_1}{2} = \frac{a_0}{2} + a_1$, so our tuple is covered.

Next, let us consider $r = 1$. Then we have $d - a_1 = -a_0 + 2a_1 + a_2 - a_1 = -a_0 + a_1 + a_2 \in \{a_2, 2a_2\}$. Since $-a_0 + a_1 + a_2 < 2a_2$, we must have $-a_0 + a_1 + a_2 = a_2 \iff a_0 = a_1$. We have $a_3 = -a_0 + a_1 + \frac{a_2}{2} = \frac{a_2}{2} < a_2$, which is a contradiction.

Finally, let us consider $r = 0$. Then we have $d - a_0 = -a_0 + 2a_1 + a_2 - a_0 = -2a_0 + 2a_1 + a_2 \in \{a_2, 2a_2\}$. If $-2a_0 + 2a_1 + a_2 = a_2$, then again we have $a_0 = a_1$ and $a_3 < a_2$, which is a contradiction. Therefore, $-2a_0 + 2a_1 + a_2 = 2a_2 \iff -2a_0 + 2a_1 = a_2$. We have $a_3 = -a_0 + a_1 + \frac{a_2}{2} = -a_0 + a_1 + a_1 - a_0 = 2a_1 - 2a_0 = a_2$ and $d = -a_0 + 2a_1 + a_2 = -a_0 + 2a_1 - 2a_0 + 2a_1 = 4a_1 - 3a_0$. Our tuple is $\big(a_0, a_1, -2a_0 + 2a_1, -2a_0 + 2a_1, 4a_1 - 3a_0\big)$. We have $2 \mid a_2, a_3$, therefore $2 \mid d \iff 2 \mid a_0$. So we have $2 \mid a_0, a_2, a_3$, which contradicts well-formedness.

$\square$

The following lemma removes the cases from Type I and Type II, which were otherwise a potential source of sporadic solutions.

Lemma 5.2. *(Same as Lemma 3.2.) Let the ordered tuple (a_0, a_1, a_2, a_3, d) satisfying $I = a_t + a_3$ or $I = \frac{a_t}{2} + a_3$ or $a_t + \frac{a_3}{2} = I$, where $t \in \{0, 1, 2\}$, be solid. Then, either $I = a_i + a_j$, for some $i, j \in \{0, 1, 2\}$ with $i \neq j$, or $I = a_0 + \frac{a_1}{2}$ or $I = a_1 + \frac{a_0}{2}$.*

Proof. Let us first consider $I = a_t + a_3$. Then $d = a_u + a_v$, where $u, v \in \{0, 1, 2\}$ and $u, v \neq t$. Condition (iv) gives $d - a_j \vdots a_3 \iff a_u + a_v - a_j \vdots a_3$. Since a_3 is the greatest of the weights and $d - a_j > 0$, we have $a_u + a_v - a_j = a_3$. So, $I = a_0 + a_1 + a_2 - a_j$. If $j \in \{0, 1, 2\}$, then the proposition is clearly satisfied, as the tuple is covered by $I = a_p + a_q$, for some p, q. This leaves the case $j = 3$. The proposition is clearly satisfied when $a_3 = a_2$. So, we must have $a_3 > a_2$. Therefore, we must also have $a_3 = \frac{a_u + a_v}{2} < \frac{a_3 + a_3}{2} = a_3$, which is a contradiction.

Now, let us consider $I = \frac{a_t}{2} + a_3$. Then $d = a_u + a_v + \frac{a_t}{2}$, where $u, v \in \{0, 1, 2\}$ and $u, v \neq t$. Condition (iv) gives $d - a_j \vdots a_3 \iff a_u + a_v + \frac{a_t}{2} - a_j \vdots a_3$. Since a_3 is the largest of the weights, we have two possibilities.

1) $a_u + a_v + \frac{a_t}{2} - a_j = 2a_3$. We will easily arrive in contradictions when considering $j = 3, u, v, t$. Since these are all the possibilities, this case vanishes.

2) $a_u + a_v + \frac{a_t}{2} - a_j = a_3$. Then, $I = \frac{a_t}{2} + a_3 = a_u + a_v + a_t - a_j$. Cases $j = u, v, t$ are clearly covered. Therefore, $j = 3$ and we have $a_3 = \frac{2a_u + 2a_v + a_t}{4}$. Also, $d = a_u + a_v + \frac{a_t}{2} = 2a_3$. The tuple is ordered $\implies a_2 \geqslant a_1 \geqslant a_0$. Let us use this to limit a_3. We have $a_3 \leqslant \frac{5a_2}{4} < 2a_2$. Let us split into two cases: $a_3 > a_2$ and $a_3 = a_2$.

2.1) $a_3 > a_2$. Let us use condition (iv), $d - a_r \vdots a_2 \iff a_u + a_v + \frac{a_t}{2} - a_r \vdots a_2$. We have $r \neq 3$ because $d - a_3 = a_3$ and $a_2 < a_3 < 2a_2$. We know $a_2 \geqslant a_u, a_v, a_t$. Either:
$r = u$ and $d - a_r = a_u + a_v + \frac{a_t}{2} - a_r = a_v + \frac{a_t}{2} \leqslant \frac{3}{2}a_2 < 2a_2$, or
$r = v$ and $d - a_r = a_u + \frac{a_t}{2} \leqslant \frac{3}{2}a_2 < 2a_2$, or
$r = t$ and $d - a_r = a_u + a_v + \frac{a_t}{2} - a_r = a_u + a_v - \frac{a_t}{2} \leqslant 2a_2 - \frac{a_t}{2} < 2a_2$.
So, in all cases $0 < d - a_r < 2a_2$. Since $d - a_r \vdots a_2$, we have $d - a_r = a_2 \iff a_u + a_v + \frac{a_t}{2} - a_r = a_2$. Previously we showed $a_3 = a_u + a_v + \frac{a_t}{2} - a_3$. Since $a_3 > a_2$, we must have $a_r > a_3$, which is impossible.

2.2) $a_3 = a_2$. We have $I = a_t + \frac{a_2}{2}$. If $t \neq 2$, then the claim holds by Lemma 3.1. This leaves the case $t = 2$. We have $d = a_u + a_v + \frac{a_t}{2} = a_0 + a_1 + \frac{a_2}{2} = 2a_3 = 2a_2$. This gives $\frac{a_2}{2} = \frac{a_0 + a_1}{3}$. Our tuple becomes $\big(a_0, a_1, \frac{2}{3}(a_0 + a_1), \frac{2}{3}(a_0 + a_1), \frac{4}{3}(a_0 + a_1)\big)$. We have $I = \frac{3}{2}a_2 = a_0 + a_1$.

Finally, let us consider $I = a_t + \frac{a_3}{2}$. Then $d = a_u + a_v + \frac{a_3}{2}$, where $u, v \in \{0, 1, 2\}$ and $u, v \neq t$. Condition (iv) gives $d - a_j \vdots a_3 \iff a_u + a_v + \frac{a_3}{2} - a_j \vdots a_3$. Since a_3 is the largest of the weights, we have $0 < a_u + a_v + \frac{a_3}{2} - a_j < \frac{5a_3}{2} < 3a_3$. There are two possibilities: either $d - a_j = 2a_3$ or $d - a_j = a_3$. We will consider them separately.

1) $a_u + a_v + \frac{a_3}{2} - a_j = 2a_3$. We will immediately arrive at contradictions when considering $j = 3, u, v$. This leaves the case $j = t$. We have $\frac{3}{2}a_3 = a_u + a_v - a_t$, giving $\frac{a_3}{2} = \frac{a_u + a_v - a_t}{3}$. So, $d = a_u + a_v + \frac{a_3}{2} = \frac{4a_u + 4a_v - a_t}{3}$ and $I = \frac{a_u + a_v + 2a_t}{3}$. Let us use condition (iv) with $i = 2$. It states $d - a_r = a_u + a_v + \frac{a_3}{2} - a_r = \frac{4a_u + 4a_v - a_t - 3a_r}{3} \vdots a_2$. We have $0 < d - a_r < \frac{4a_2 + 4a_2}{3} = \frac{8}{3}a_2 < 3a_2$. So, either $d - a_r = 2a_2$ or $d - a_r = a_2$. Let us consider them separately.

1.1) $d - a_r = 2a_2$. This is equivalent to $\frac{4a_u + 4a_v - a_t - 3a_r}{3} = 2a_2$, giving $4a_u + 4a_v - a_t - 3a_r = 6a_2$. Cases $r = u, v, 1, 2, 3$ immediately give contradictions. This means $r = t = 0$. We have $4a_u + 4a_v - 4a_t = 6a_2$. Since $u, v \in \{1, 2\}$, this gives $2a_1 + 2a_2 - 2a_0 = 3a_2 \iff 2a_1 - 2a_0 = a_2$.

It follows that $\frac{a_3}{2} = \frac{a_u+a_v-a_t}{3} = \frac{-a_0+a_1+a_2}{3} = \frac{-a_0+a_1+2a_1-2a_0}{3} = \frac{-3a_0+3a_1}{3} = -a_0 + a_1$. This means $a_3 = 2a_1 - 2a_0 = a_2$, and $d = a_1 + a_2 + \frac{a_3}{2} = a_1 + 2a_1 - 2a_0 + a_1 - a_0 = -3a_0 + 4a_1$. Since $2 \mid a_2, a_3$, well-formedness requires $2 \mid a_0$. We have $2 \mid a_0, a_2, a_3$, which is a contradiction.
1.2) $d - a_r = a_2$. This is equivalent to $\frac{4a_u+4a_v-a_t-3a_r}{3} = a_2$, giving $4a_u + 4a_v - a_t - 3a_r = 3a_2$. Let us consider all the cases separately. There are up to $3! \cdot 4 = 24$ different cases.
If $t = 0$, we have $u, v \in \{1, 2\}$, and the equation becomes $-a_0+4a_1+4a_2-3a_r = 3a_2 \iff a_2 = a_0 - 4a_1 + 3a_r$. It follows that $r \in \{2, 3\}$. If $r = 2$, we have $a_2 = 2a_1 - \frac{a_0}{2}$, and $I = \frac{2a_0+a_1+a_2}{3} = \frac{a_0}{2} + a_1$, as required. If $r = 3$, we have $a_2 = a_0 - 4a_1 + 3a_3 = a_0 - 4a_1 - 2a_0 + 2a_1 + 2a_2 = -a_0 - 2a_1 + 2a_2$. This is equivalent to $a_2 = a_0 + 2a_1$. We have $I = \frac{2a_0+a_1+a_2}{3} = a_0 + a_1$.
If $t = 1$, we have $u, v \in \{0, 2\}$, and the equation becomes $4a_0 - a_1 + 4a_2 - 3a_r = 3a_2 \iff a_2 = -4a_0+a_1+3a_r$. If $r = 3$, we have $a_2 = -4a_0+a_1+2a_0-2a_1+2a_2 = -2a_0-a_1+2a_2 \iff a_2 = 2a_0+a_1$. We have $I = \frac{a_0+2a_1+a_2}{3} = a_0+a_1$. If $r = 2$, we have $a_2 = 2a_0 - \frac{a_1}{2}$ and $I = \frac{a_0+2a_1+a_2}{3} = a_0 + \frac{a_1}{2}$. If $r = \{0, 1\}$, we have $a_2 = -4a_0 + a_1 + 3a_r$ and $I = \frac{a_0+2a_1+a_2}{3} = -a_0 + a_1 + a_r$. If $r = 0$, we have $a_2 = -a_0 + a_1$ and $I = a_1 = a_0 + a_2$. If $r = 1$, we have $a_2 = -4a_0 + 4a_1$, and $a_3 = \frac{2a_0-2a_1+2a_2}{3} = -6a_0 + 6a_1$, and $d = \frac{4a_0-a_1+4a_2}{3} = -4a_0 + 5a_1$. Since $2 \mid a_2, a_3$, we must have $2 \mid d$, which implies $2 \mid a_1$. We have $2 \mid a_1, a_2, a_3$, which contradicts well-formedness.
If $t = 2$, we have $u, v \in \{0, 1\}$, and the equation becomes $4a_0 + 4a_1 - a_2 - 3a_r = 3a_2 \iff a_2 = a_0 + a_1 - \frac{3a_r}{4}$. We have $a_3 = \frac{2a_0+2a_1-2a_2}{3}$ and $I = \frac{a_0+a_1+2a_2}{3}$. If $r = 3$, we have $a_2 = a_0 + a_1 + \frac{-a_0-a_1+a_2}{2} = \frac{a_0+a_1+a_2}{2} \iff a_2 = a_0 + a_1$ and $I = a_0 + a_1$. If $r = 2$, we have $a_2 = \frac{4a_0+4a_1}{7}$ and $a_3 = \frac{2a_0+2a_1-2a_2}{3} = \frac{14a_0+14a_1-14a_2}{21} = \frac{14a_0+14a_1-8a_0-8a_1}{21} = \frac{2a_0+2a_1}{7} < a_2$, which is a contradiction. If $r = 1$, we have $a_2 = a_0 + \frac{a_1}{4}$ and $a_3 = \frac{2a_0+2a_1-2a_2}{3} = \frac{4a_0+4a_1-4a_2}{6} = \frac{4a_0+4a_1-4a_2}{6} = \frac{1}{2}a_1 < a_1$, which is a contradiction. If $r = 0$, we have $a_2 = a_0 + a_1 - \frac{3a_r}{4} = \frac{a_0}{4} + a_1$ and $a_3 = \frac{2a_0+2a_1-2a_2}{3} = \frac{4a_0+4a_1-4a_2}{6} = \frac{a_0}{2} < a_0$, which is a contradiction.

2) $a_u + a_v + \frac{a_3}{2} - a_j = a_3$. Then $\frac{a_3}{2} = a_u + a_v - a_j$. We have $I = a_t + \frac{a_3}{2} = a_t + a_u + a_v - a_j$. Cases $j = u, v, t$ are clearly covered. Therefore, $j = 3$ and we have $\frac{a_3}{2} = \frac{a_u+a_v}{3}$. Also, $d = a_u + a_v + \frac{a_3}{2} = \frac{4a_u+4a_v}{3} = 2a_3$. The tuple is ordered. Let us use this to put bounds on a_3. We have $a_3 \leqslant \frac{4a_2}{3} < 2a_2$. Let us split into two cases: $a_3 > a_2$ and $a_3 = a_2$.
2.1) $a_3 > a_2$. Let us use condition (iv), $d - a_r \mathrel{\vdots} a_2 \iff \frac{4a_u+4a_v-3a_r}{3} \mathrel{\vdots} a_2$. We have $r \neq 3$ because $d - a_3 = a_3$ and $a_2 < a_3 < 2a_2$. We know $a_2 \geqslant a_u, a_v, a_t$. Either:
$r = u$ and $d - a_r \leqslant \frac{5}{3}a_2 < 2a_2$, or
$r = v$ and $d - a_r \leqslant \frac{5}{3}a_2 < 2a_2$, or
$r = t$ and if $r \geqslant 1$, we have $d - a_r \leqslant \frac{5}{3}a_2 < 2a_2$.
So, in all these three cases $0 < d - a_r < 2a_2$. Since $d - a_r \mathrel{\vdots} a_2$, we have $d - a_r = a_2$. We will include here the case, when $r = t = 0$ and $d - a_r = a_2$. So, there are two options: either $d - a_r = a_2$, or $r = t = 0$ and $d - a_r = 2a_2$. We will consider them separately.
If $r = t = 0$ and $d - a_r = 2a_2$, then $d - a_0 = \frac{4a_u+4a_v-3a_0}{3} = 2a_2$. Since $u, v \in \{1, 2\}$, we have $\frac{-3a_0+4a_1+4a_2}{3} = 2a_2 \iff -3a_0 + 4a_1 + 4a_2 = 6a_2 \iff a_2 = \frac{-3a_0+4a_1}{2}$ and $\frac{a_3}{2} = \frac{a_1+a_2}{3} = \frac{2a_1+2a_2}{6} = \frac{2a_1-3a_0+4a_1}{6} = -\frac{a_0}{2} + a_1$. We have $I = a_0 + \frac{a_3}{2} = \frac{a_0}{2} + a_1$, as required.
If it does not hold that $r = t = 0$ and $d - a_r = 2a_2$, then we must have $d - a_r = a_2 \iff 2a_3 - a_r = a_2 \iff 2a_3 = a_2 + a_r$. We cannot have $r = 3$, since we assumed $a_3 > a_2$. Therefore, $2a_3 > a_2 + a_r$, which contradicts the above.
2.2) $a_3 = a_2$. We have $I = a_t + \frac{a_2}{2}$. If $t \neq 2$, then the claim holds by Lemma 3.1. This leaves the case $t = 2$, implying $u, v \in \{0, 1\}$. We have $d = a_u + a_v + \frac{a_3}{2} = a_0 + a_1 + \frac{a_2}{2} = 2a_3 = 2a_2$. This gives $\frac{a_2}{2} = \frac{a_0+a_1}{3}$. Our tuple becomes $\big(a_0, a_1, \frac{2}{3}(a_0 + a_1), \frac{2}{3}(a_0 + a_1), \frac{4}{3}(a_0 + a_1)\big)$. We have $I = \frac{3}{2}a_2 = a_0 + a_1$, as required.

□

The following lemma justifies the definition of the series for classes 4 and 5 in Definition 2.5.

Lemma 5.3. *(Same as Lemma 3.4.) Let the ordered tuple (a_0, a_1, a_2, a_3, d) satisfying $a_t+\frac{a_u}{2} = I$, where $t, u \in \{0, 1\}$ and $t \neq u$, be solid. Then $a_3 = \frac{u}{2} + a_2$ and $d = 2a_3 = a_u + 2a_2$.*

Proof. Let us first consider $I = \frac{a_0}{2} + a_1$. Then, $d = \sum a_s - I = \frac{a_0}{2} + a_2 + a_3$. From condition (iv), we have $d - a_j = \frac{a_0}{2} + a_2 + a_3 - a_j \mathrel{\vdots} a_3$. The frame contains condition (iii), which requires $d-a_j > 0$. If $d-a_j \geqslant 2a_3$, then $\frac{a_0}{2}+a_2+a_3-a_j \geqslant 2a_3 \iff \frac{a_0}{2}+a_2 \geqslant a_j+a_3$, which is false, since $\frac{a_0}{2} < a_j$ and $a_2 \leqslant a_3$. Therefore, $d - a_j = a_3$. We get $\frac{a_0}{2} + a_2 + a_3 - a_j = a_3 \iff \frac{a_0}{2} + a_2 = a_j$.

From this we see $a_j > a_2$, so it leaves the only option $j = 3$. Therefore, $a_3 = \frac{a_0}{2} + a_2$ and $d = \sum a_i - I = a_0 + 2a_2 = 2a_3$, as claimed.

Now, let us consider $I = a_0 + \frac{a_1}{2}$. Then, $d = \sum a_s - I = \frac{a_1}{2} + a_2 + a_3$. From condition (iv), we have $d - a_j = \frac{a_1}{2} + a_2 + a_3 - a_j \,\vdots\, a_3$.

If $d - a_j \geqslant 3a_3$, then $\frac{a_1}{2} + a_2 + a_3 - a_j \geqslant 3a_3 \iff \frac{a_1}{2} + a_2 \geqslant a_j + 2a_3$, which is false, since $\frac{a_1}{2} < a_j$ and $a_2 < 2a_3$.

If $d - a_j = 2a_3$, then $\frac{a_1}{2} + a_2 + a_3 - a_j = 2a_3 \iff \frac{a_1}{2} + a_2 = a_j + a_3$. If $j \geqslant 1$, then $a_1 + a_3 > \frac{a_1}{2} + a_2$, which is a contradiction. Therefore, $j = 0$, and we get $\frac{a_1}{2} + a_2 = a_0 + a_3 \implies a_3 = \frac{a_1}{2} + a_2 - a_0$ and $d = \sum a_i - I = a_0 + a_1 + a_2 + \frac{a_1}{2} + a_2 - a_0 - a_0 - \frac{a_1}{2} = -a_0 + a_1 + 2a_2$. Now, let us use condition (iv) with $i = 2$. We have $d - a_j \,\vdots\, a_2 \iff a_1 - a_0 - a_j \,\vdots\, a_2$.
If $j = 0$, we have $a_1 - a_0 - a_0 = a_1 - 2a_0 \,\vdots\, a_2$. If $a_2 = a_0$, then $a_0 = a_1 = a_2$ which implies $a_0 = a_1 = a_2 = 1$. This contradicts the equation $I = a_0 + \frac{a_1}{2}$ and the fact that I must be an integer. So, $a_2 > a_0$, and we have $-a_2 < -a_0 \leqslant a_1 - 2a_0 < a_1 \leqslant a_2$. So, $a_1 = 2a_0$. From this we get $a_3 = \frac{a_1}{2} + a_2 - a_0 = a_2$ and $d = a_0 + 2a_2$. Our tuple becomes $\big(a_0, 2a_0, a_2, a_2, a_0 + 2a_2\big)$. Let us use condition (iv) with $i = 0$. We must have $d - a_j = a_0 + 2a_2 - a_j \,\vdots\, a_0 \iff 2a_2 - a_j \,\vdots\, a_0$. Regardless of what j we choose, we must have $2a_2 \,\vdots\, a_0$. If a_0 is odd, then $a_2 \,\vdots\, a_0$ which implies $a_2 = a_0$ and we arrive at a contradiction as before. Therefore a_0 is even. So, $a_2 \,\vdots\, \frac{a_0}{2}$. If $\frac{a_0}{2} > 1$, then $\gcd(a_0, a_1, a_2) = \gcd(a_0, 2a_0, a_2) \geqslant \frac{a_0}{2} > 1$ which contradicts being well-formed. So $\frac{a_0}{2} = 1 \implies a_0 = 2$. Our tuple becomes $\big(a_0, 2a_0, a_2, a_2, 2a_2 + a_0\big) = \big(2, 4, a, a, 2a + 2\big)$. Well-formedness requires $a \mid 2a + 2 \implies a \mid 2 \implies a \in \{1, 2\}$. This contradicts the tuple being ordered, as a must satisfy $a \geqslant 4$.
If $j = 1$, we have $a_1 - a_0 - a_1 = -a_0 \,\vdots\, a_2 \implies a_0 = a_2$, and we arrive at a contradiction as before.
If $j = 2$, we have $a_1 - a_0 - a_2 \,\vdots\, a_2 \iff a_1 - a_0 \,\vdots\, a_2$. Since $0 \leqslant a_1 - a_0 < a_1 \leqslant a_2$, we must have $a_0 = a_1$. We have $I = \frac{a_0}{2} + a_1$. Using the first part of this proof, we get $d = 2a_3$ and $a_3 = \frac{a_0}{2} + a_2 = \frac{a_1}{2} + a_2$, as required.
If $j = 3$, we have $a_1 - a_0 + a_0 - \frac{a_1}{2} - a_2 \,\vdots\, a_2 \iff \frac{a_1}{2} \,\vdots\, a_2$, which cannot hold.
These four cases together show that the case $d - a_j = 2a_3$ has no solid tuples.

If $d - a_j = a_3$, then $\frac{a_1}{2} + a_2 + a_3 - a_j = a_3 \iff \frac{a_1}{2} + a_2 = a_j$. From this we see $a_j > a_2$, leaving the only option $j = 3$. Therefore, $a_3 = \frac{a_1}{2} + a_2$ and $d = \sum a_i - I = a_0 + a_1 + a_2 + \frac{a_1}{2} + a_2 - a_0 - \frac{a_1}{2} = a_1 + 2a_2 = 2a_3$, as claimed.

□

The following proposition, which is one of the main results of this paper, shows it suffices only to consider conditions (i)-(iv) to classify the hypersurfaces.

Proposition 5.4. *(Same as Proposition 4.1.) A solid tuple is valid.*

Proof. This proposition says that if a tuple satisfies conditions (i)-(iv), it also satisfies (v) and (vi).

Having proven that a solid tuple is colourful (Proposition 3.3), it suffices to consider the six classes given in Definition 2.4.

Classes 1-3. $I = a_t + a_u$, where $t, u \leqslant 2$ and $t < u$. Let us define $r, s \in \{0, 1, 2\}$ such that $r < s$ and r, s, t, u are pairwise different. Then $d = a_r + a_s$. Conditions (v) and (vi) hold for $(i, j) = (r, s)$, since $d - a_r = a_s \,\vdots\, a_s$ and $d - a_s = a_r \,\vdots\, a_r$. Condition (v) holds for all $(i, j) \neq (t, u)$, since for $(i, j) \neq (r, s), (t, u)$, we have $\gcd(i, j) = 1$. Let us show this. Let $v \in \{t, u\}$, $p \in \{r, s\}$ and $q \in \{r, s\}$ with $p \neq q$. We have $\gcd(a_v, a_p) \mid a_v, a_p$, implying $\gcd(a_v, a_p) \mid d = a_p + a_q$, giving $\gcd(a_v, a_p) \mid a_v, a_p, a_q$. So, well-formedness requires $\gcd(a_v, a_p) = 1$. So, we have to show condition (v) for $(i, j) = (t, u)$ and condition (vi) for $(i, j) \neq (r, s)$.

Let us use condition (iv) with $i = t$. Depending on j, one of the following must hold:

- $d - a_r = a_s \,\vdots\, a_t$. We have $a_t \mid a_t, a_s$, so well-formedness requires $a_t \mid d$.
- $d - a_s = a_r \,\vdots\, a_t$. Analogously $a_t \mid d$.
- $d - a_t \,\vdots\, a_t \iff d \,\vdots\, a_t$.
- $d - a_u \,\vdots\, a_t$.

We have that either $d \,\vdots\, a_t$ or $d - a_u \,\vdots\, a_t$, which means conditions (v) and (vi) hold for $(i, j) = (t, u)$, and therefore condition (v) holds for all (i, j). Analogously for $i = u$, we get that either $d \,\vdots\, a_u$

or $d - a_t \vdots a_u$. Let us examine the cases separately. We have to check that condition (vi) holds for $(i,j) \neq (r,s),(t,u)$. So we choose $i \in \{r,s\}$ and $j \in \{t,u\}$.

1) $d \vdots a_t$ and $d \vdots a_u$. Clearly condition (vi) holds.

2) $d - a_u \vdots a_t$ and $d \vdots a_u$. Condition (vi) holds when $j = u$. So, we just have to check condition (vi) for $(i,j) = (r,t)$ and $(i,j) = (s,t)$.
If $(i,j) = (r,t)$, then using (b)-part of condition (vi), we have $(k,l) = (s,u)$. We have $d - a_k \vdots a_r$ and $d - a_l \vdots a_t$, so the condition holds. If $(i,j) = (s,t)$, then using (b)-part of condition (vi), we have $(k,l) = (r,u)$. We have $d - a_k \vdots a_s$ and $d - a_l \vdots a_t$, so the condition holds.

3) $d \vdots a_t$ and $d - a_t \vdots a_u$. Analogous to 2).

4) $d - a_u \vdots a_t$ and $d - a_t \vdots a_u$. Let us show condition (vi) holds for general (i,j), with $i \in \{r,s\}$ and $j \in \{t,u\}$. Let us define $p \in \{r,s\}$ with $p \neq i$, and $v \in \{t,u\}$ with $v \neq j$. Then $d = a_i + a_p$. Using condition (vi) (b), we have $(k,l) = (p,v)$. We have $d - a_k = d - a_p = a_i \vdots a_i$ and $d - a_l \vdots a_j$, since $d - a_t \vdots a_u$ and $d - a_u \vdots a_t$. So, conditions (v) and (vi) hold.

Classes 4-5. $I = a_t + \frac{a_u}{2}$, where $t,u \in \{0,1\}$ with $t \neq u$. We have $d = \frac{a_u}{2} + a_2 + a_3 = a_u + 2a_2 = 2a_3$ and $a_3 = \frac{a_u}{2} + a_2$. Since $d = 2a_3$, we have $d \vdots 2$.

Let us use condition (iv) with $i = t$. Depending on j, one of the following must hold:

- $d - a_t \vdots a_t$, implying $d \vdots a_t$.
- $d - a_u = 2a_2 \vdots a_t$. Either $a_2 \vdots a_t$ implying due to well-formedness that $d \vdots a_t \iff a_u \vdots a_t$, meaning $a_t = 1$ and $d \vdots a_t$, or $a_t \vdots 2$ and $a_2 \vdots \frac{a_t}{2}$. In the latter case we have, due to well-formedness, that $a_u \vdots \frac{a_t}{2}$. Since $\frac{a_t}{2} \mid a_2, a_t, a_u$, we must have that $\frac{a_t}{2} = 1 \iff a_t = 2$. We have $d \vdots 2 = a_t$. So, in both cases $d \vdots a_t$.
- $d - a_2 \vdots a_t$.
- $d - a_3 = a_3 \vdots a_t$, implying $d \vdots a_t$.

Let us use condition (iv) with $i = u$. Depending on j, one of the following must hold:

- $d - a_t \vdots a_u$.
- $d - a_u \vdots a_u$, implying $d \vdots a_u$.
- $d - a_2 = a_u + a_2 \vdots a_u$, implying $a_2 \vdots a_u$, and well-formedness implies $d \vdots a_u$.
- $d - a_3 = a_3 \vdots a_u$, implying $d \vdots a_u$.

So, we always have either $d \vdots a_t$ or $d - a_2 \vdots a_t$, and $d \vdots a_u$ or $d - a_t \vdots a_u$. The former guarantees that conditions (v) and (vi) are satisfied with $(i,j) = (t,2)$, the latter solves $(i,j) = (t,u)$. The case $(i,j) = (u,2)$ is satisfied, since $d - a_u = 2a_2 \vdots a_2$. Cases where $j = 3$ are satisfied, since $d \vdots a_3$. So, conditions (v) and (vi) are satisfied.

Class 6. The ordered tuple $\big(a_0, a_1, a_2, a_3, d\big)$ has the form $\big(I - k, I + k, a, a + k, 2a + I + k\big)$. We have $d = a_1 + 2a_2 = a_0 + 2a_3$.

Let us use condition (iv) with $i = 0$. Depending on j, one of the following must hold:

- $d - a_0 \vdots a_0$, implying $d \vdots a_0$.
- $d - a_1 = 2a_2 \vdots a_0$. This means that either $a_2 \vdots a_0$, with well-formedness requiring $d \vdots a_0$, or $a_0 \vdots 2$ and $a_2 \vdots \frac{a_0}{2}$, with well-formedness requiring $d = a_1 + 2a_2 \vdots \frac{a_0}{2} \iff a_1 \vdots \frac{a_0}{2}$. Well-formedness requires $\frac{a_0}{2} = 1 \iff a_0 = 2$. Since $d = a_0 + 2a_3$, we have $d \vdots 2 = a_0$. So, in both cases $d \vdots a_0$.
- $d - a_2 \vdots a_0$.
- $d - a_3 = a_3 + a_0 \vdots a_0 \iff a_3 \vdots a_0$, implying $d \vdots a_0$.

Let us use condition (iv) with $i = 1$. Depending on j, one of the following must hold:

- $d - a_0 = 2a_3 \vdots a_1$. This means that either $a_3 \vdots a_1$, with well-formedness requiring $d \vdots a_1$, or $a_1 \vdots 2$ and $a_3 \vdots \frac{a_1}{2}$, with well-formedness requiring $d = a_0 + 2a_3 \vdots \frac{a_1}{2} \iff a_0 \vdots \frac{a_1}{2}$. Well-formedness requires $\frac{a_1}{2} = 1 \iff a_1 = 2$. Since $d = a_1 + 2a_2$, we have $d \vdots 2 = a_1$. So, in both cases $d \vdots a_1$.
- $d - a_1 \vdots a_1$, implying $d \vdots a_1$.
- $d - a_2 = a_2 + a_1 \vdots a_1 \iff a_2 \vdots a_1$, implying $d \vdots a_1$.
- $d - a_3 \vdots a_1$.

So, we always have either $d \vdots a_0$ or $d - a_2 \vdots a_0$, and $d \vdots a_1$ or $d - a_3 \vdots a_1$. The former guarantees $(i,j) = (0,2)$ is satisfied, the latter that $(i,j) = (1,3)$ is satisfied. Cases $(i,j) = (1,2)$ and $(i,j) = (0,3)$ are already satisfied, since $d - a_1 = 2a_2 \vdots a_2$ and $d - a_0 = 2a_3 \vdots a_3$. This leaves the cases $(i,j) \in \{(0,1),(2,3)\}$. Let us first deal with $(i,j) = (2,3)$. We will show that $\gcd(a_2, a_3) = 1$. We have $\gcd(a_2, a_3) = \gcd(a, a + k) = \gcd(a, k) \mid k$. Well-formedness

requires $\gcd(a_2, a_3) \mid d = 2a_2 + a_1 \iff \gcd(a_2, a_3) \mid a_1$. So, $\gcd(a_2, a_3) \mid a_1, a_2, a_3$, meaning $\gcd(a_2, a_3) = 1$. This shows condition (v) is satisfied with $(i, j) = (2, 3)$. Let us show condition (vi) (b) is satisfied. We have $(k, l) = (0, 1)$. We get $d - a_k = 2a_3 \vdots a_3$ and $d - a_l = 2a_2 \vdots a_2$. So, condition (vi) is satisfied. It is only left to show conditions (v) and (vi) are satisfied for $(i, j) = (0, 1)$. If $d \vdots a_0$ or $d \vdots a_1$, this is clear. So, it remains to consider the case $d - a_2 \vdots a_0$ and $d - a_3 \vdots a_1$. Let us first show $\gcd(a_0, a_1) = 1$. First, let us show that a_0 and a_1 are both odd (with the current premises). It is easy to see that a_0 and a_1 have the same parity. If a_0 and a_1 are both even, then well-formedness requires a_2 and a_3 to be odd. Well-formedness requires d to be even. We must have $d - a_2 \vdots a_0 \vdots 2$, but $d - a_2$ is odd. Contradiction! So, a_0 and a_1 are odd. We have $\gcd(a_0, a_1) = \gcd(I - k, I + k) = \gcd(I - k, 2k) = \gcd(I - k, k) = \gcd(I, k)$. Well-formedness requires $d = 2a + k + I \vdots \gcd(I, k) \iff 2a \vdots \gcd(I, k) \iff a \vdots \gcd(I, k)$. So, well-formedness requires $\gcd(a_0, a_1) = 1$. So, we have but to check condition (vi) (b). We have $(k, l) = (2, 3)$. We get $d - a_k \vdots a_0$ and $d - a_l \vdots a_1$. This shows conditions (v) and (vi) are satisfied. □

We know previously that every valid tuple belongs to one of the infinite series given in 2.5. The following theorem shows that to find all the valid tuples, it suffices to check the quasi-smoothness conditions for just a single tuple from each series. This is the main theorem of the paper.

Theorem 5.5. *(Same as Theorem 4.2.) A solid tuple is valid. Whenever a tuple is valid, all the ordered tuples in its corresponding series, given in Definition 2.5, are valid.*

Proof. We say a tuple is ordered when $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 < d$, with a strict inequality for the degree.

As shown in Proposition 4.1, it suffices to consider solid tuples instead of valid. Let us consider the six classes given in Definition 2.4 separately.

Class 1. We have $I = a_0 + a_1$ and $d = a_2 + a_3$. Our tuple is $A = \big(a_0, a_1, a_2, a_3, a_2 + a_3\big)$. A tuple in its series is given by $B = \big(b_0, b_1, b_2, b_3, b_2 + b_3\big) = \big(a_0, a_1, a_2 + xm, a_3 + ym, a_2 + a_3 + (x + y)m\big)$ with $m = \operatorname{lcm}(a_0, a_1)$, for some integers x, y, given the weights are ordered. Let us show that when our given tuple A is solid, then the tuple B is also solid.

First, let us check well-formedness (conditions (i) and (ii)). We have $\gcd(b_2, b_3) \mid g$, where g is the degree of the tuple B. For all other pairs, we have $\gcd(b_i, b_j) = \gcd(a_i, a_j) \mid d$. Since one of i, j belongs to $\{0, 1\}$, we have $\gcd(a_i, a_j) \mid m$. So, $\gcd(b_i, b_j) = \gcd(a_i, a_j) \mid g = d + (x + y)m$. Also, for (i, j, k), one of them must belong to $\{0, 1\}$, so $\gcd(b_i, b_j, b_k) = \gcd(a_i, a_j, a_k) = 1$.

Condition (iii) holds, since we require the tuple to be ordered, with a strict inequality for the degree.

It is left to check condition (iv). It holds naturally for $i \in \{2, 3\}$. For $i \in \{0, 1\}$, we have $b_i = a_i$, so we can cancel out the terms with an m: $g - b_j \vdots b_i \iff d - a_j \vdots a_i$. Since we defined the tuple A such that condition (iv) holds, it holds for B also.

Classes 2-3. $I = a_t + a_2$, where $t \in \{0, 1\}$. Let us define $v \in \{0, 1\}$ with $v \neq t$. Then $d = a_v + a_3$. Our tuple is $A = \big(a_0, a_1, a_2, a_3, a_t + a_3\big)$. A tuple in its series is given by $B = \big(b_0, b_1, b_2, b_3, b_t + b_3\big) = \big(a_0, a_1, a_2, a_3 + xm, a_2 + a_3 + xm\big)$ with $m = \operatorname{lcm}(a_0, a_1, a_2)$, for some integer x, given the weights are ordered. Let us show that when our given tuple A is solid, then the tuple B is also solid.

First, let us check well-formedness (conditions (i) and (ii)). For all pairs (i, j), we have $\gcd(b_i, b_j) = \gcd(a_i, a_j) \mid d$. Since $\gcd(a_i, a_j) \mid m$, we have $\gcd(b_i, b_j) \mid g$. Since $\gcd(b_i, b_j) = \gcd(a_i, j)$, for all (i, j), we must have that for all triples (i, j, k) it holds that $\gcd(b_i, b_j, b_k) = \gcd(a_i, a_j, a_k) = 1$.

Condition (iii) holds, since tuple B is defined to be ordered.

It is left to check condition (iv). It holds naturally for $i = 3$. For $i \in \{0, 1, 2\}$, we have $b_i = a_i$, so we can cancel out the terms with an m: $g - b_j \vdots b_i \iff d - a_j \vdots a_i$. So, condition (iv) holds and the tuple B is solid.

Classes 4-5. $I = a_t + \frac{a_u}{2}$, where $t, u \in \{0, 1\}$ with $t \neq u$. We have $d = \frac{a_u}{2} + a_2 + a_3 = a_u + 2a_2 = 2a_3$ and $a_3 = \frac{a_u}{2} + a_2$. Since $d = 2a_3$, we have $d \vdots 2$. Our tuple is $A = \big(a_0, a_1, a_2, a_3, d\big)$. A tuple in its series is given by $B = \big(b_0, b_1, b_2, b_3, g\big) = \big(a_0, a_1, a_2 + xm, a_3 + xm, d + 2xm\big)$ with

$m = \operatorname{lcm}(a_0, a_1, a_2)$, for some integer x, given the weights are ordered. Let us show that when our given tuple A is solid, then the tuple B is also solid.

First, let us check well-formedness (conditions (i) and (ii)). We have $\gcd(b_2, b_3) \mid g = 2b_3$, where g is the degree of the tuple B. For all other pairs, we have $\gcd(b_i, b_j) = \gcd(a_i, a_j) \mid d$. Since one of i, j belongs to $\{0, 1\}$, we have $\gcd(a_i, a_j) \mid m$. So, $\gcd(b_i, b_j) = \gcd(a_i, a_j) \mid g = d + 2xm$. Also, for (i, j, k), one of them must belong to $\{0, 1\}$, so $\gcd(b_i, b_j, b_k) = \gcd(a_i, a_j, a_k) = 1$.

Condition (iii) holds by definition.

It is left to check condition (iv). It holds naturally for $i \in \{2, 3\}$. For $i \in \{0, 1\}$, we have $b_i = a_i$, so we can cancel out the terms with an m: $g - b_j \vdots b_i \iff d - a_j \vdots a_i$. So, tuple B is also solid.

Class 6. The ordered tuple (a_0, a_1, a_2, a_3, d) has the form $(I - k, I + k, a, a + k, 2a + I + k)$. We have $d = a_1 + 2a_2 = a_0 + 2a_3$. Our tuple is $A = (a_0, a_1, a_2, a_3, d)$. A tuple in its series is given by $B = (b_0, b_1, b_2, b_3, g) = (a_0, a_1, a_2 + xm, a_3 + xm, d + 2xm)$ with $m = \operatorname{lcm}(a_0, a_1, k)$, for some integer x, given the weights are ordered. Let us show that when our given tuple A is solid, then the tuple B is also solid.

First, let us check well-formedness (conditions (i) and (ii)). As $\gcd(b_2, b_3) \mid k$, we have $\gcd(b_2, b_3) = \gcd(a_2, a_3) \mid d$, implying $\gcd(b_2, b_3) \mid g$. For all other pairs, we have $\gcd(b_i, b_j) = \gcd(a_i, a_j) \mid d$. For $i < j$, we must have $i \in \{0, 1\}$. So, $\gcd(b_i, b_j) = \gcd(a_i, a_j) \mid m$, and $\gcd(b_i, b_j) \mid g$. Also, for (i, j, k), one of them must belong to $\{0, 1\}$, so $\gcd(b_i, b_j, b_k) = \gcd(a_i, a_j, a_k) = 1$.

Condition (iii) holds by definition.

It is left to check condition (iv). It holds naturally for $i \in \{2, 3\}$. For $i \in \{0, 1\}$, we have $b_i = a_i$, so we can cancel out the terms with an m: $g - b_j \vdots b_i \iff d - a_j \vdots a_i$. So, tuple B is also solid.

$\square$

6. Small Index Cases.

In this section, we will give the lists of all quasi-smooth, well-formed hypersurfaces for indices $I = 1, 2, \ldots, 6$. In all these tables the parameters x and y are non-negative integers with $x \leqslant y$.

Table 1: Index 1, Infinite Series

(a_0, a_1, a_2, a_3)	d
$(2, 2x+3, 2x+3, 4x+5)$	$8x+12$

Table 2: Index 1, Sporadic Cases

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1,1,1,1)$	3	$(1,1,1,2)$	4	$(1,1,2,3)$	6
$(1,2,3,5)$	10	$(1,3,5,7)$	15	$(1,3,5,8)$	16
$(2,3,5,9)$	18	$(3,3,5,5)$	15	$(3,5,7,11)$	25
$(3,5,7,14)$	28	$(3,5,11,18)$	36	$(5,14,17,21)$	56
$(5,19,27,31)$	81	$(5,19,27,50)$	100	$(7,11,27,37)$	81
$(7,11,27,44)$	88	$(9,15,17,20)$	60	$(9,15,23,23)$	69
$(11,29,39,49)$	127	$(11,49,69,128)$	256	$(13,23,35,57)$	127
$(13,35,81,128)$	256				

Table 3: Index 2, Two-Parameter Series

(a_0, a_1, a_2, a_3)	d
$(1, 1, x+1, y+1)$	$x+y+2$

Table 4: Index 2, Infinite Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 2, x+2, x+3)$	$2x+6$	$(1, 3, 3x+3, 3x+4)$	$6x+9$
$(1, 3, 3x+4, 3x+5)$	$6x+11$	$(3, 3x+3, 3x+4, 3x+4)$	$9x+12$
$(3, 3x+4, 3x+5, 3x+5)$	$9x+15$	$(3, 3x+4, 3x+5, 6x+7)$	$12x+17$
$(3, 3x+4, 6x+7, 9x+9)$	$18x+21$	$(3, 3x+4, 6x+7, 9x+12)$	$18x+24$
$(4, 2x+5, 2x+5, 4x+8)$	$8x+20$	$(4, 2x+5, 4x+10, 6x+13)$	$12x+30$

Table 5: Index 2, Sporadic Cases

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 3, 4, 6)$	12	$(1, 4, 5, 7)$	15	$(1, 4, 5, 8)$	16
$(1, 4, 6, 9)$	18	$(1, 5, 7, 11)$	22	$(1, 6, 9, 13)$	27
$(1, 6, 10, 15)$	30	$(1, 7, 12, 18)$	36	$(1, 8, 13, 20)$	40
$(1, 9, 15, 22)$	45	$(2, 3, 4, 5)$	12	$(2, 3, 4, 7)$	14
$(3, 4, 5, 10)$	20	$(3, 4, 6, 7)$	18	$(3, 4, 10, 15)$	30
$(5, 13, 19, 22)$	57	$(5, 13, 19, 35)$	70	$(6, 9, 10, 13)$	36
$(7, 8, 19, 25)$	57	$(7, 8, 19, 32)$	64	$(9, 12, 13, 16)$	48
$(9, 12, 19, 19)$	57	$(9, 19, 24, 31)$	81	$(10, 19, 35, 43)$	105
$(11, 21, 28, 47)$	105	$(11, 25, 32, 41)$	107	$(11, 25, 34, 43)$	111
$(11, 43, 61, 113)$	226	$(13, 18, 45, 61)$	135	$(13, 20, 29, 47)$	107
$(13, 20, 31, 49)$	111	$(13, 31, 71, 113)$	226	$(14, 17, 29, 41)$	99

Table 6: Index 3, Two-Parameter Series

(a_0, a_1, a_2, a_3)	d
$(1, 2, 2x+3, 2y+3)$	$2(x+y)+6$

Table 7: Index 3, Infinite Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 1, 2, 2x+3)$	$2x+4$	$(1, 5, 10x+5, 10x+7)$	$20x+15$
$(1, 5, 10x+7, 10x+9)$	$20x+19$		

Table 8: Index 3, Sporadic Cases

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 7, 9, 13)$	27	$(1, 7, 9, 14)$	28	$(1, 9, 13, 20)$	40
$(1, 13, 22, 33)$	66	$(1, 14, 23, 35)$	70	$(1, 15, 25, 37)$	75

Table 8: Index 3, Sporadic Cases

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(5, 7, 11, 13)$	33	$(5, 7, 11, 20)$	40	$(11, 21, 29, 37)$	95
$(11, 37, 53, 98)$	196	$(13, 17, 27, 41)$	95	$(13, 27, 61, 98)$	196
$(15, 19, 43, 74)$	148				

Table 9: Index 4, Two-Parameter Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 3, 3x + 5, 3y + 5)$	$3(x + y) + 10$	$(1, 3, 3x + 5, 3y + 7)$	$3(x + y) + 12$
$(2, 2, 2x + 3, 2y + 3)$	$2(x + y) + 6$	$(1, 3, 3x + 4, 3y + 5)$	$3(x + y) + 9$

Table 10: Index 4, Infinite Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 1, 3, 3x + 5)$	$3x + 6$	$(1, 2, 2, 2x + 3)$	$2x + 4$
$(1, 2, 3, 6x + 4)$	$6x + 6$	$(1, 2, 3, 6x + 5)$	$6x + 7$
$(1, 2, 3, 6x + 7)$	$6x + 9$	$(1, 2, 3, 6x + 8)$	$6x + 10$
$(1, 7, 21x + 7, 21x + 10)$	$42x + 21$	$(1, 7, 21x + 10, 21x + 13)$	$42x + 27$
$(1, 7, 21x + 14, 21x + 17)$	$42x + 35$	$(1, 7, 21x + 17, 21x + 20)$	$42x + 41$
$(2, 3, 3x + 4, 3x + 5)$	$6x + 10$	$(2, 3, 3x + 5, 3x + 6)$	$6x + 12$
$(2, 4, 2x + 5, 2x + 7)$	$4x + 14$	$(2, 6, 6x + 9, 6x + 11)$	$12x + 24$
$(3, 5, 15x + 5, 15x + 6)$	$30x + 15$	$(3, 5, 15x + 10, 15x + 11)$	$30x + 25$
$(3, 5, 15x + 11, 15x + 12)$	$30x + 27$	$(3, 5, 15x + 16, 15x + 17)$	$30x + 37$
$(6, 6x + 9, 6x + 11, 6x + 11)$	$18x + 33$	$(6, 6x + 11, 12x + 20, 18x + 27)$	$36x + 60$
$(6, 6x + 11, 12x + 20, 18x + 33)$	$36x + 66$		

Table 11: Index 4, Sporadic Cases

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 10, 13, 19)$	39	$(1, 10, 13, 20)$	40	$(1, 13, 19, 29)$	58
$(1, 14, 21, 31)$	63	$(1, 19, 32, 48)$	96	$(1, 20, 33, 50)$	100
$(1, 21, 35, 52)$	105	$(2, 7, 10, 15)$	30	$(2, 9, 12, 17)$	36
$(5, 6, 8, 9)$	24	$(5, 6, 8, 15)$	30	$(9, 11, 12, 17)$	45
$(10, 13, 25, 31)$	75	$(11, 17, 20, 27)$	71	$(11, 17, 24, 31)$	79
$(11, 31, 45, 83)$	166	$(13, 14, 19, 29)$	71	$(13, 14, 23, 33)$	79
$(13, 23, 51, 83)$	166				

Table 12: Index 5, Two-Parameter Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 4, 4x + 5, 4y + 7)$	$4(x + y) + 12$	$(1, 4, 4x + 7, 4y + 9)$	$4(x + y) + 16$
$(2, 3, 6x + 5, 6y + 7)$	$6(x + y) + 12$	$(2, 3, 6x + 7, 6y + 7)$	$6(x + y) + 14$

Table 12: Index 5, Two-Parameter Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(2, 3, 6x + 7, 6y + 11)$	$6(x + y) + 18$		

Table 13: Index 5, Infinite Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 1, 4, 4x + 7)$	$4x + 8$	$(1, 2, 3, 6x + 5)$	$6x + 6$
$(1, 2, 3, 6x + 7)$	$6x + 8$	$(1, 3, 4, 12x + 5)$	$12x + 8$
$(1, 3, 4, 12x + 9)$	$12x + 12$	$(1, 3, 4, 12x + 13)$	$12x + 16$
$(1, 9, 36x + 9, 36x + 13)$	$72x + 27$	$(1, 9, 36x + 13, 36x + 17)$	$72x + 35$
$(1, 9, 36x + 27, 36x + 31)$	$72x + 63$	$(1, 9, 36x + 31, 36x + 35)$	$72x + 71$
$(3, 7, 42x + 7, 42x + 9)$	$84x + 21$	$(3, 7, 42x + 23, 42x + 25)$	$84x + 53$
$(3, 7, 42x + 35, 42x + 37)$	$84x + 77$	$(3, 7, 42x + 37, 42x + 39)$	$84x + 81$

Table 14: Index 5, Sporadic Cases

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 13, 17, 25)$	51	$(1, 13, 17, 26)$	52	$(1, 17, 25, 38)$	76
$(1, 25, 42, 63)$	126	$(1, 26, 43, 65)$	130	$(1, 27, 45, 67)$	135
$(6, 7, 9, 10)$	27	$(11, 13, 19, 25)$	63	$(11, 25, 37, 68)$	136
$(13, 19, 41, 68)$	136				

Table 15: Index 6, Two-Parameter Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 5, 5x + 6, 5y + 9)$	$5(x + y) + 15$	$(1, 5, 5x + 7, 5y + 8)$	$5(x + y) + 15$
$(1, 5, 5x + 7, 5y + 9)$	$5(x + y) + 16$	$(1, 5, 5x + 8, 5y + 8)$	$5(x + y) + 16$
$(1, 5, 5x + 8, 5y + 12)$	$5(x + y) + 20$	$(1, 5, 5x + 9, 5y + 11)$	$5(x + y) + 20$
$(1, 5, 5x + 9, 5y + 12)$	$5(x + y) + 21$	$(2, 4, 4x + 5, 4y + 5)$	$4(x + y) + 10$
$(2, 4, 4x + 5, 4y + 7)$	$4(x + y) + 12$	$(2, 4, 4x + 7, 4y + 7)$	$4(x + y) + 14$
$(2, 4, 4x + 7, 4y + 9)$	$4(x + y) + 16$	$(3, 3, 3x + 4, 3y + 5)$	$3(x + y) + 9$
$(3, 3, 3x + 5, 3y + 7)$	$3(x + y) + 12$		

Table 16: Index 6, Infinite Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 1, 5, 5x + 9)$	$5x + 10$	$(1, 2, 4, 4x + 5)$	$4x + 6$
$(1, 2, 4, 4x + 7)$	$4x + 8$	$(1, 2, 5, 10x + 8)$	$10x + 10$
$(1, 2, 5, 10x + 9)$	$10x + 11$	$(1, 2, 5, 10x + 13)$	$10x + 15$
$(1, 2, 5, 10x + 14)$	$10x + 16$	$(1, 3, 3, 3x + 5)$	$3x + 6$
$(1, 3, 5, 15x + 7)$	$15x + 10$	$(1, 3, 5, 15x + 8)$	$15x + 11$
$(1, 3, 5, 15x + 12)$	$15x + 15$	$(1, 3, 5, 15x + 13)$	$15x + 16$

Table 16: Index 6, Infinite Series

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 3, 5, 15x + 17)$	$15x + 20$	$(1, 3, 5, 15x + 18)$	$15x + 21$
$(1, 4, 5, 20x + 6)$	$20x + 10$	$(1, 4, 5, 20x + 7)$	$20x + 11$
$(1, 4, 5, 20x + 11)$	$20x + 15$	$(1, 4, 5, 20x + 12)$	$20x + 16$
$(1, 4, 5, 20x + 16)$	$20x + 20$	$(1, 4, 5, 20x + 17)$	$20x + 21$
$(1, 4, 5, 20x + 21)$	$20x + 25$	$(1, 4, 5, 20x + 22)$	$20x + 26$
$(1, 11, 55x + 11, 55x + 16)$	$110x + 33$	$(1, 11, 55x + 16, 55x + 21)$	$110x + 43$
$(1, 11, 55x + 22, 55x + 27)$	$110x + 55$	$(1, 11, 55x + 27, 55x + 32)$	$110x + 65$
$(1, 11, 55x + 33, 55x + 38)$	$110x + 77$	$(1, 11, 55x + 38, 55x + 43)$	$110x + 87$
$(1, 11, 55x + 44, 55x + 49)$	$110x + 99$	$(1, 11, 55x + 49, 55x + 54)$	$110x + 109$
$(2, 3, 3, 6x + 4)$	$6x + 6$	$(2, 3, 3, 6x + 7)$	$6x + 9$
$(2, 3, 4, 12x + 5)$	$12x + 8$	$(2, 3, 4, 12x + 7)$	$12x + 10$
$(2, 3, 4, 12x + 9)$	$12x + 12$	$(2, 3, 4, 12x + 11)$	$12x + 14$
$(2, 3, 4, 12x + 13)$	$12x + 16$	$(2, 3, 4, 12x + 15)$	$12x + 18$
$(2, 5, 5x + 8, 5x + 9)$	$10x + 18$	$(2, 5, 5x + 9, 5x + 10)$	$10x + 20$
$(2, 8, 4x + 9, 4x + 13)$	$8x + 26$	$(2, 10, 20x + 15, 20x + 19)$	$40x + 40$
$(2, 10, 20x + 25, 20x + 29)$	$40x + 60$	$(5, 7, 35x + 8, 35x + 9)$	$70x + 23$
$(5, 7, 35x + 14, 35x + 15)$	$70x + 35$	$(5, 7, 35x + 28, 35x + 29)$	$70x + 63$
$(5, 7, 35x + 29, 35x + 30)$	$70x + 65$	$(8, 4x + 9, 4x + 11, 4x + 13)$	$12x + 35$
$(9, 3x + 11, 3x + 14, 6x + 19)$	$12x + 47$		

Table 17: Index 6, Sporadic Cases

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 16, 21, 31)$	63	$(1, 16, 21, 32)$	64	$(1, 21, 31, 47)$	94
$(1, 22, 33, 49)$	99	$(1, 31, 52, 78)$	156	$(1, 32, 53, 80)$	160
$(1, 33, 55, 82)$	165	$(2, 13, 18, 27)$	54	$(2, 15, 20, 29)$	60
$(3, 7, 8, 12)$	24	$(7, 10, 15, 19)$	45	$(11, 19, 29, 53)$	106
$(13, 15, 31, 53)$	106				

As I grows the lists get larger and larger – the lists grow as the cube of the index and the computation time grows as the fifth power of I. This requires a lot of computation for greater indices. The natural way to deal with this is to program the algorithm. The source-code for the main idea of the algorithm is given in Appendix B.

Appendix A. Tables

The following tables are taken from [11, Appendix B].

Table 18 and Table 19 contain one-parameter infinite series and sporadic cases respectively of values of $(a_0, a_1, a_2, a_3, d, I)$. The last columns represent the cases in [57] from which the sextuples $(a_0, a_1, a_2, a_3, d, I)$ originate[1]. The parameter n is any positive integer.

[1] Note that sometimes a sextuple $(a_0, a_1, a_2, a_3, d, I)$ originates from several cases in [57].

Table 18: Infinite series

(a_0, a_1, a_2, a_3)	d	I	Source
$(1, 3n-2, 4n-3, 6n-5)$	$12n-9$	n	VII.2(3)
$(1, 3n-2, 4n-3, 6n-4)$	$12n-8$	n	II.2(2)
$(1, 4n-3, 6n-5, 9n-7)$	$18n-14$	n	VII.3(1)
$(1, 6n-5, 10n-8, 15n-12)$	$30n-24$	n	III.1(4)
$(1, 6n-4, 10n-7, 15n-10)$	$30n-20$	n	III.2(2)
$(1, 6n-3, 10n-5, 15n-8)$	$30n-15$	n	III.2(4)
$(1, 8n-2, 12n-3, 18n-5)$	$36n-9$	$2n$	IV.3(3)
$(2, 6n-3, 8n-4, 12n-7)$	$24n-12$	$2n$	II.2(4)
$(2, 6n+1, 8n+2, 12n+3)$	$24n+6$	$2n+2$	II.2(1)
$(3, 6n+1, 6n+2, 9n+3)$	$18n+6$	$3n+3$	II.2(1)
$(7, 28n-22, 42n-33, 63n-53)$	$126n-99$	$7n-2$	XI.3(14)
$(7, 28n-18, 42n-27, 63n-44)$	$126n-81$	$7n-1$	XI.3(14)
$(7, 28n-17, 42n-29, 63n-40)$	$126n-80$	$7n+1$	X.3(1)
$(7, 28n-13, 42n-23, 63n-31)$	$126n-62$	$7n+2$	X.3(1)
$(7, 28n-10, 42n-15, 63n-26)$	$126n-45$	$7n+1$	XI.3(14)
$(7, 28n-9, 42n-17, 63n-22)$	$126n-44$	$7n+3$	X.3(1)
$(7, 28n-6, 42n-9, 63n-17)$	$126n-27$	$7n+2$	XI.3(14)
$(7, 28n-5, 42n-11, 63n-13)$	$126n-26$	$7n+4$	X.3(1)
$(7, 28n-2, 42n-3, 63n-8)$	$126n-9$	$7n+3$	XI.3(14)
$(7, 28n-1, 42n-5, 63n-4)$	$126n-8$	$7n+5$	X.3(1)
$(7, 28n+2, 42n+3, 63n+1)$	$126n+9$	$7n+4$	XI.3(14)
$(7, 28n+3, 42n+1, 63n+5)$	$126n+10$	$7n+6$	X.3(1)
$(2, 2n+1, 2n+1, 4n+1)$	$8n+4$	1	II.3(4)
$(3, 3n, 3n+1, 3n+1)$	$9n+3$	2	III.5(1)
$(3, 3n+1, 3n+2, 3n+2)$	$9n+6$	2	II.5(1)
$(3, 3n+1, 3n+2, 6n+1)$	$12n+5$	2	XVIII.2(2)
$(3, 3n+1, 6n+1, 9n)$	$18n+3$	2	VII.3(2)
$(3, 3n+1, 6n+1, 9n+3)$	$18n+6$	2	II.2(2)
$(4, 2n+1, 2n+1, 4n)$	$8n+4$	2	V.3(4)
$(4, 2n+3, 4n+6, 6n+7)$	$12n+18$	2	XII.3(17)
$(6, 6n-1, 12n-4, 18n-9)$	$36n-12$	4	VII.3(2)
$(6, 6n-1, 12n-4, 18n-3)$	$36n-6$	4	IV.3(1)
$(6, 6n+3, 6n+5, 6n+5)$	$18n+15$	4	III.5(1)
$(8, 4n+5, 4n+7, 4n+9)$	$12n+23$	6	XIX.2(2)
$(9, 3n+5, 3n+8, 6n+7)$	$12n+23$	6	XIX.2(2)

Table 19: Sporadic cases

(a_0, a_1, a_2, a_3)	d	I	Source	(a_0, a_1, a_2, a_3)	d	I	Source
$(1, 3, 5, 8)$	16	1	VIII.3(5)	$(2, 3, 5, 9)$	18	1	II.2(3)

Table 19: Sporadic cases

(a_0, a_1, a_2, a_3)	d	I	Source	(a_0, a_1, a_2, a_3)	d	I	Source
$(3,3,5,5)$	15	1	I.19	$(3,5,7,11)$	25	1	X.2(3)
$(3,5,7,14)$	28	1	VII.4(4)	$(3,5,11,18)$	36	1	VII.3(1)
$(5,14,17,21)$	56	1	XI.3(8)	$(5,19,27,31)$	81	1	X.3(3)
$(5,19,27,50)$	100	1	VII.3(3)	$(7,11,27,37)$	81	1	X.3(4)
$(7,11,27,44)$	88	1	VII.3(5)	$(9,15,17,20)$	60	1	VII.6(3)
$(9,15,23,23)$	69	1	III.5(1)	$(11,29,39,49)$	127	1	XIX.2(2)
$(11,49,69,128)$	256	1	X.3(1)	$(13,23,35,57)$	127	1	XIX.2(2)
$(13,35,81,128)$	256	1	X.3(2)	$(1,3,4,6)$	12	2	I.3
$(1,4,6,9)$	18	2	IV.3(3)	$(1,6,10,15)$	30	2	I.4
$(2,3,4,7)$	14	2	IX.3(1)	$(3,4,5,10)$	20	2	II.3(2)
$(3,4,6,7)$	18	2	VII.3(10)	$(3,4,10,15)$	30	2	II.2(3)
$(5,13,19,22)$	57	2	X.3(3)	$(5,13,19,35)$	70	2	VII.3(3)
$(6,9,10,13)$	36	2	VII.3(8)	$(7,8,19,25)$	57	2	X.3(4)
$(7,8,19,32)$	64	2	VII.3(3)	$(9,12,13,16)$	48	2	VII.6(2)
$(9,12,19,19)$	57	2	III.5(1)	$(9,19,24,31)$	81	2	XI.3(20)
$(10,19,35,43)$	105	2	XI.3(18)	$(11,21,28,47)$	105	2	XI.3(16)
$(11,25,32,41)$	107	2	XIX.3(1)	$(11,25,34,43)$	111	2	XIX.2(2)
$(11,43,61,113)$	226	2	X.3(1)	$(13,18,45,61)$	135	2	XI.3(14)
$(13,20,29,47)$	107	2	XIX.3(1)	$(13,20,31,49)$	111	2	XIX.2(2)
$(13,31,71,113)$	226	2	X.3(2)	$(14,17,29,41)$	99	2	XIX.2(3)
$(5,7,11,13)$	33	3	X.3(3)	$(5,7,11,20)$	40	3	VII.3(3)
$(11,21,29,37)$	95	3	XIX.2(2)	$(11,37,53,98)$	196	3	X.3(1)
$(13,17,27,41)$	95	3	XIX.2(2)	$(13,27,61,98)$	196	3	X.3(2)
$(15,19,43,74)$	148	3	X.3(1)	$(9,11,12,17)$	45	4	XI.3(20)
$(10,13,25,31)$	75	4	XI.3(14)	$(11,17,20,27)$	71	4	XIX.3(1)
$(11,17,24,31)$	79	4	XIX.2(2)	$(11,31,45,83)$	166	4	X.3(1)
$(13,14,19,29)$	71	4	XIX.3(1)	$(13,14,23,33)$	79	4	XIX.2(2)
$(13,23,51,83)$	166	4	X.3(2)	$(11,13,19,25)$	63	5	XIX.2(2)
$(11,25,37,68)$	136	5	X.3(1)	$(13,19,41,68)$	136	5	X.3(2)
$(11,19,29,53)$	106	6	X.3(1)	$(13,15,31,53)$	106	6	X.3(2)
$(11,13,21,38)$	76	7	X.3(1)				

Appendix B. Source Code

The source code for the program to classify hypersurfaces of index I is given here. It is written in a simple form, without focusing on simplifying the output or adding series from the tables. It is given for the purpose of illustrating how the algorithm can be coded. The full program and its source code, which is much longer, is available from the author. The program is written in Haskell.

```
-- Classifying quasi-smooth well-formed weighted hypersurfaces.
-- Erik Paemurru

data Tuple = Quint Int Int Int Int Int Int Int deriving( Eq,Ord,Show )
--                  (a0, a1, a2, a3, dd, mm, cc)
```

```
-- dd - degree
-- mm - series modulo-number
-- cc - series class-number

main = do
  putStr ("Enter index, for which to solve:\n" ++ "Index = ")
  strL <- getLine
  mapM_ putStrLn (map show (solve (read strL :: Int)))

-- The 'solve' function classifies the hypersurfaces. The input 'ii' is the index.
-- The result is a list of 7-tuples in the form given above. Using the definition
-- of infinite series, it is straightforward to write down the series, given a
-- tuple of this form. Tuples from the tables must also be added, which is not
-- implemented in this sample code.
solve ii = map (filter tests) (makeTuples ii)

lcm3 a b c = lcm a (lcm b c)
gcd3 a b c = gcd a (gcd b c)

-- div   a b gives a/b rounded down.
-- divUp a b gives a/b rounded up.
divUp a b = -((-a) `div` b)

makeTuples ii = [makeClass cc ii | cc <- [1..6]]

makeClass cc ii
  | cc == 1  = concat [makeClassDefWei a0 (ii - a0)  0 (lcm  a0    (ii - a0)) cc
      ii | a0 <- [1..(ii `div` 2)]]
  | cc == 2  = concat [makeClassDefWei a0 a1 (ii - a0) (lcm3 a0 a1 (ii - a0)) cc
      ii | a0 <- [1..(ii `div` 2)], a1 <- [a0..(ii-a0-1)]]
  | cc == 3  = concat [makeClassDefWei a0 a1 (ii - a1) (lcm3 a0 a1 (ii - a1)) cc
      ii | a1 <- [2..(ii `div` 2)], a0 <- [1..(a1-1)]]
-- for c==4, we put k = a1/2. Also, we assume m is twice as small as in the paper.
  | cc == 4  = concat [makeClassDefWei (ii-k) (2*k) 0 (lcm (ii-k) k) cc ii | k <-
      [(max (ii `divUp` 3) 1)..(ii-1)]]
-- for c==5, we put k = a0/2. Also, we assume m is twice as small as in the paper.
  | cc == 5  = concat [makeClassDefWei (2*k) (ii-k) 0 (lcm (2*k) k) cc ii | k <-
      [1..((ii `divUp` 3)-1)]]
  | cc == 6  = concat [makeClassDefWei (ii-k) (ii+k) k (lcm3 (ii-k) (ii+k) k) cc
      ii | k <- [1..(ii-1)]]

-- makeClassDefWei - make class given defining weights and mm.
makeClassDefWei b0 b1 b2 mm cc ii
-- for c==1, we have (b0,b1,b2) = (a0,a1,*)
  | cc == 1  = [Quint b0 b1 a2 a3 (a2 + a3) mm cc | a2 <- [b1..(b1+mm-1)], a3 <- [
      a2..(a2+mm-1)]]
-- for c==2, we have (b0,b1,b2) = (a0,a1,a2)
  | cc == 2  = [Quint b0 b1 b2 a3 (b1 + a3) mm cc | a3 <- [b2..(b2+mm-1)]]
-- for c==3, we have (b0,b1,b2) = (a0,a1,a2)
  | cc == 3  = [Quint b0 b1 b2 a3 (b0 + a3) mm cc | a3 <- [b2..(b2+mm-1)]]
-- for c==4, we have (b0,b1,b2) = (a0,a1,*)
  | cc == 4  = [Quint b0 b1 a2 (a2 + (b1 `div` 2)) (2*(a2 + (b1 `div` 2))) mm cc |
       a2 <- [b1..(b1+mm-1)]]
-- for c==5, we have (b0,b1,b2) = (a0,a1,*)
  | cc == 5  = [Quint b0 b1 a2 (a2 + (b0 `div` 2)) (2*(a2 + (b0 `div` 2))) mm cc |
       a2 <- [b1..(b1+mm-1)]]
-- for c==6, we have (b0,b1,b2) = (a0,a1,k)
  | cc == 6  = [Quint b0 b1 a2 (a2 + b2) (b1 + 2*a2) mm cc | a2 <- [b1..(b1+mm-1)
      ]]

-- tests - well-formedness and quasi-smoothness conditions.
tests tuple = and[test j tuple | j <- [1..3]]

test j (Quint a0 a1 a2 a3 dd mm cc)
  | j == 1  = and[dd `mod` (gcd ai aj) == 0 | (ai,aj) <- pairs]
  | j == 2  = and[gcd3 ai aj ak == 1 | (ai,aj,ak) <- triples]
-- j==3 is condition (iv)
```

```
  | j == 3  = and[or[(dd - aj) `mod` ai == 0 | aj <- weights] | ai <- weights]
  where
  weights = [a0,a1,a2,a3]
  pairs = [(a0,a1),(a0,a2),(a0,a3),(a1,a2),(a1,a3),(a2,a3)]
  triples = [(a0,a1,a2),(a0,a1,a3),(a0,a2,a3),(a1,a2,a3)]
```

.

References

[1] V. Alexeev, V. Nikulin, *Del Pezzo and K3 surfaces*
Mathematical Society of Japan Memoirs **15** (2006)

[2] C. Araujo, *Kähler–Einstein metrics for some quasi-smooth log del Pezzo surfaces*
Transactions of the American Mathematical Society **354** (2002), 4303–3312

[3] M. Artin, D. Mumford, *Some elementary examples of unirational varieties which are not rational*
Proceedings of the London Mathematical Society **25** (1972), 75–95

[4] T. Aubin, *Equations du type Monge–Ampère sur les variétés Kähleriennes compactes*
Bulletin des Sciences Mathématique **354** (2002), 4303–3312

[5] G. Belousov, *The maximal number of singular points on log del Pezzo surfaces*
Journal of Mathematical Sciences, the University of Tokyo **16** (2009), 231–238

[6] C. Boyer, K. Galicki, M. Nakamaye, *On the geometry of Sasakian–Einstein 5-manifolds*
Mathematische Annalen **325** (2003), 485–524

[7] C. Birkar, P. Cascini, C. Hacon, J. McKernan, *Existence of minimal models for varieties of log general type*
Journal of the American Mathematical Society **23** (2010), 405–468

[8] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces*
Geometric and Functional Analysis, **18** (2008), 1118–1144

[9] I. Cheltsov, J Park, C. Shramov, *Exceptional del Pezzo hypersurfaces*
Journal of Geometric Analysis **20** (2010), 787–816

[10] I. Cheltsov, C. Shramov, *Log canonical thresholds of smooth Fano threefolds*
Russian Mathematical Surveys **63** (2008), 859–958

[11] I. Cheltsov, C. Shramov, *Del Pezzo zoo*
Experimental Mathematics, to appear

[12] H. Clemens, P. Griffiths, *The intermediate Jacobian of the cubic threefold*
Annals of Mathematics **95** (1972), 73–100

[13] A. Corti, *Factorizing birational maps of threefolds after Sarkisov*
Journal of Algebraic Geometry **4** (1995), 223–254

[14] J.-P. Demailly, J. Kollár, *Semi-continuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds*
Annales Scientifiques de l'École Normale Supérieure **34** (2001), 525–556

[15] S. Donaldson, *Lower bounds on the Calabi functional*
Journal of Differential Geometry **70** (2005), 453–472

[16] S. Donaldson, *Conjectures in Kähler geometry*
Strings and geometry, American Mathematical Society, Providence, RI (2005), 71–78

[17] S. Donaldson, *Stability, birational transformations and the Kähler-Einstein problem*
arXiv:1007.4220 (2010)

[18] S. Donaldson, *b-Stability and blow-ups*
arXiv:1107.1699 (2011)

[19] S. Donaldson, *Discussion of the Kähler–Einstein problem*
preprint, http://www2.imperial.ac.uk/~skdona/KENOTES.PDF

[20] A. Elagin, *Exceptional sets on del Pezzo surfaces with one log-terminal singularity*
Mathematical Notes **82**, No. 1–2 (2007), 33–46

[21] A. Futaki, *An obstruction to the existence of Einstein–Kähler metrics*
Inventiones Mathematicae **73** (1983), 437–443

[22] J. Gauntlett, D. Martelli, J. Sparks, S.-T. Yau, *Obstructions to the existence of Sasaki-Einstein metrics*
Communications in Mathematical Physics **273** (2007) 803–827

[23] P. Hacking, Y. Prokhorov, *Smoothable del Pezzo surfaces with quotient singularities*
Compositio Mathematica **146** (2010), 169–192

[24] F. Hidaka, K. Watanabe, *Normal Gorenstein surfaces with ample anti-canonical divisor*
Tokyo Journal of Mathematics **4** (1981), 319–330

[25] A. Iano-Fletcher, *Working with weighted complete intersections*
L.M.S. Lecture Note Series **281** (2000), 101–173

[26] A. Ishii, K. Ueda, *The special McKay correspondence and exceptional collection* arXiv:1104.2381 (2011)

[27] V. Iskovskikh, Yu. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem* Matematical Sbornik **86** (1971), 140–166

[28] V. Iskovskikh, Yu. Prokhorov, *Fano varieties* Encyclopaedia of Mathematical Sciences **47** (1999) Springer, Berlin

[29] J. Johnson, J. Kollár, *Kähler–Einstein metrics on log del Pezzo surfaces in weighted projective* 3*-spaces* Annales de l'Institut Fourier **51** (2001), 69–79

[30] A. Kasprzyk, M. Kreuzer, B. Nill, *On the combinatorial classification of toric log del Pezzo surfaces* LMS Journal of Computation and Mathematics **13** (2010), 33–46

[31] Y. Kawamata, *Derived categories of toric varieties* Michigan Mathematical Journal **54** (2006), 517–536

[32] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model problem* Advanced Studies in Pure Mathematics **10** (1987), 283–360

[33] S. Keel, J. McKernan, *Rational curves on quasi-projective surfaces* Memoirs of the American Mathematical Society **669** (1999)

[34] H. Kojima, *Del Pezzo surfaces of rank one with unique singular points* Japan Journal of Mathematics **25** (1999), 343–374

[35] J. Kollár, *Singularities of pairs* Proceedings of Symposia in Pure Mathematics **62** (1997), 221–287

[36] J. Kollár, S. Mori, *Birational geometry of algebraic varieties* Cambridge University Press (1998)

[37] S. Kudryavtsev, *Classification of exceptional log del Pezzo surfaces with* $\delta = 1$ Izvestia: Mathematics, **67** (2003), 461–497.

[38] M. Lübke, *Stability of Einstein-Hermitian vector bundles* Manuscripta Mathematica **42** (1983), 245–257

[39] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne* Nagoya Mathematical Journal **11** (1957), 145–150

[40] M. Miyanishi, D. Q. Zhang, *Gorenstein log del Pezzo surfaces of rank one* Journal of Algebra **118** (1988), 63–84

[41] D. Phong, J. Sturm, *Lectures on stability and constant scalar curvature* Current developments in mathematics, International Press (2009), 101–176

[42] J. Ross, R. Thomas, *A study of Hilbert-Mumford criterion of stability of projective varieties* Journal of Algebraic Geometry **16** (2007), 201–255

[43] J. Ross, R. Thomas, *An obstruction to the existence of constant scalar curvature Kähler metrics* Journal of Differential Geometry **72** (2006), 429–466

[44] J. Ross, R. Thomas, *Weighted Bergman kernels on orbifolds* Journal of Differential Geometry **88** (2011) 87–108

[45] J. Ross, R. Thomas, *Weighted projective embeddings, stability of orbifolds and constant scalar curvature Kähler metrics* Journal of Differential Geometry **88** (2011) 109–159

[46] V. Shokurov, *Three-fold log flips* Russian Academy of Sciences, Izvestiya Mathematics **40** (1993), 95–202

[47] V. Shokurov, *Complements on surfaces* Journal of Mathematical Sciences **102** (2000), 3876–3932

[48] J. Song, *The* α*-invariant on toric Fano threefolds* American Journal of Mathematics **127** (2005), 1247–1259

[49] C. Spotti, *Degenerations of Kähler–Einstein Fano manifolds* Ph.D. thesis, Imperial College, London (UK), 2012

[50] G. Tian, *On Kähler–Einstein metrics on certain Kähler manifolds with* $c_1(M) > 0$ Inventiones Mathematicae **89** (1987), 225–246

[51] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class* Inventiones Mathematicae **101** (1990), 101–172

[52] G. Tian, *Kähler–Einstein metrics with positive scalar curvature* Inventiones Mathematicae **130** (1997), 1–37

[53] G. Tian, S.-T. Yau, *Kähler–Einstein metrics metrics on complex surfaces with* $C_1 > 0$ Communications in Mathematical Physics **112** (1987), 175–203

[54] X. Wang, X. Zhu, *Kähler–Ricci solitons on toric manifolds with positive first Chern class* Advances in Mathematics **188** (2004), 87–103

[55] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I* Communications on Pure and Applied Mathematics **31** (1978), 339–411

[56] S.-T. Yau, *Review on Kähler–Einstein metrics in algebraic geometry*
Israel Mathematics, Conference Proceedings **9** (1996), 433–443

[57] S. Yau, Y. Yu, *Classification of 3-dimensional isolated rational hypersurface singularities with* $\mathbb{C}^*$*-action*
Rocky Mountain Journal of Mathematics **35** (2005), 1795–1802

[58] T. Urabe, *On singularities on degenerate Del Pezzo surfaces of degree* 1, 2
Proceedings of Symposia in Pure Mathematics **40** (1983), 587–590

[59] D.-Q. Zhang, *Logarithmic del Pezzo surfaces of rank one with contractible boundaries*
Osaka Journal of Mathematics **25** (1988), 461–497

Erik Paemurru
School of Mathematics
University of Edinburgh
e.paemurru@sms.ed.ac.uk